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The Optimal Control Algorithms in Systems with Different Rates of Motion

The basic equations and formulas allow to obtain a decision applying the method of moments.

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Published by

Science Publishing Group

1 Rockefeller Plaza, 10th and 11th Floors,
New York, NY 10020 U.S.A.

http://www.sciencepublishinggroup.com

ISBN: 978-1-940366-72-2

Preface

Methods of optimal control, namely, the moments method and the small parameter method are rarely used for the solution of economic problems. Using these methods to economic processes will make it possible to take into account certain influencing factors and their effects, as well as possible to evaluate the changes in the processes.

Application of the method of moments to problems of optimal control of linear, quasi-linear systems are considered in N. N. Krasovskiy N. N. [86], Butkovskiy A. G. [16], Albrecht E. G. [2, 3], Egorov A. I. [37]. Moreover, in those considering systems are used diversely optimized functions of time, linearity and norm.

The method of moments can often help to find the kind of control actions in a closed analytical form [16], and in cases where this is not possible, gives a single computational procedure for constructing the exact or approximate numerical solution of the problem. The complexity of this procedure does not depend on the number of control actions, it depends only on the order of the equation and the nature of the Eigen functions of problem. Application of the method of moments to the economic problems of optimal control singularly perturbed systems in domestic and foreign sources practically does not meet.

Small perturbations in problems of optimal control can be introduced artificially, and then perturbation theory appears as a method of research of the original problem [19]. In this sense, it can be applied to the study of the properties of the main of the trajectories and modes of development of the economic system.

The book is dedicated to two aspects: first aspect is proposed approximate method of decomposition of the original problem of optimal control, which allows us to formulate it in the form of the problem of moment. It is a new direction in relation to the system under study in the theory of control. Second aspect is the studing of the results of dynamic processes optimal control of the economy.

Contents

Chapter 1

Decomposition in a Singularly Perturbed System

1.1 Integral Manifolds and Separation Movements

Consider the problem of separation singularly perturbed equations of the dynamics of the controlled system

$$\dot{x} = A_1(t)x + A_2(t)z + B_1(t)u + f_1(t),$$
$$\mu\dot{z} = A_3(t)x + A_4(t)z + B_2(t)u + f_2(t),$$
$$x(t_0) = x^0, \quad z(t_0) = z^0$$

or

$$\dot{y}(t,\mu) = A(t,\mu)y(t,\mu) + B(t,\mu)u(t) + f(t,\mu), \tag{1.1.1}$$
$$y(t_i) = y^i, \quad i = 0,1,$$

where

$$A(t,\mu) = \begin{pmatrix} A_1(t) & A_2(t) \\ \dfrac{1}{\mu}A_3(t) & \dfrac{1}{\mu}A_4(t) \end{pmatrix}, \; y(t,\mu) = \begin{pmatrix} x(t,\mu) \\ z(t,\mu) \end{pmatrix}, \; B(t,\mu) = \begin{pmatrix} B_1(t) \\ \dfrac{B_2(t)}{\mu} \end{pmatrix},$$

$f(t,\mu) = \begin{pmatrix} f_1(t) \\ \dfrac{1}{\mu} f_2(t) \end{pmatrix}$, $x(t) \in R^n$, $z(t) \in R^m$ - vectors of slow and fast coordinate system (1.1.1) $u(t) \in R^r$ - control vector function; μ - a small positive parameter, $t \in [t_0, t_1]$, vector functions $f_1(t) \in R^n$, $f_2(t) \in R^m$ - characterize a constant external force.

Suppose that the following conditions are met:

I. Matrices $A_1(t)$, $A_2(t)$, $A_3(t)$, $A_4(t)-$ defined uniformly bounded and uniformly continuous with their derivatives with $t \in [t_0, t_1]$.

II. Eigenvalues value of matrix $A_4(t)$ submits to an inequality

$$\operatorname{Re}\lambda_i(t) \le -\gamma < 0, \quad (i = \overline{1,m}), \tag{1.1.2}$$

where $\gamma > 0$ - some a constant, $t \in [t_0, t_1]$.

At $u = 0$, instead of (1.1.1) we receive uncontrollable system

$$\begin{aligned} \dot{x} &= A_1(t)x + A_2(t)z + f_1(t), \\ \mu\dot{z} &= A_3(t)x + A_4(t)z + f_2(t). \end{aligned} \tag{1.1.3}$$

If the conditions I, II, then the system (1.2.2) is an integral manifold [98]

$$z = H(t,\mu)x + \tilde{z}\,. \tag{1.1.4}$$

Then the slow movement on the integral manifold (1.1.4) describes the system

$$\dot{x} = (A_1(t) + A_2(t)H(t,\mu))x + A_2(t)\tilde{z} + f_1(t), \tag{1.1.5}$$

where the matrix $H = H(t,\mu)$ and the vector $\tilde{z} = \tilde{z}(t,\mu)$ are determined from the equation

$$\begin{aligned} \mu\dot{H} &= -\mu H A_1(t) + A_4(t)H + A_3 - \mu H A_3(t)H, \\ \mu\dot{\tilde{z}} &= (A_4(t) - \mu H(t,\mu)A_2(t))\tilde{z} + (f_2(t) - \mu H(t,\mu)\, f_1(t)). \end{aligned} \tag{1.1.6}$$

Making the system (1.1.3) replacement $z = H(t,\mu)x + \tilde{z}(t,\mu) + \eta$ can be divided into fast and slow movements

$$\begin{aligned} \overline{x} &= (A_1(t) + A_2(t)H(t,\mu))x + A_2(t)\tilde{z} + f_1(t) + A_2(t)\eta, \\ \mu\dot{\eta} &= (A_4(t) - \mu H(t,\mu)A_2(t)\eta. \end{aligned} \tag{1.1.7}$$

This procedure is described in [44] when dividing of slow and fast movements uncontrollable system.

Direct application of this approach to system (1.1.1) is not possible. This is due to the following reasons: when considering problem of optimal control, we are interested primarily controllability of the system and the selection of the control function. A selection of the control function is carried out by various criteria of optimality and is associated with other problems.

Depending formulation of the problem, the above method can be applied only for the intermediate results of the general problem, as is done in [44].

If we act in the same way as the way the system (1.1.1), then the second equation (1.1.6) will be even additional term, which contains the control function, which has not yet been determined.

In the first equation (1.1.6) contains a small nonlinearity, it is necessary to set the initial condition in the beginning it is necessary to establish the existence and uniqueness of the solution of this equation. If the initial value problem for this equation is solvable, then there is another question that relates to the passage to the limit $\mu \to 0$ in the area of the boundary layer [18].

Given these observations outlined here offer significantly modified approach integral manifold, which allows for "a complete separation of" slow and fast coordinate system (1.1.1) and get a new system with traffic separation, which has all the properties of the original system. And this in turn makes it possible to formulate the optimal control problem under the constraint (1.1.1) in the form of the problem of moment [42], which is a new step towards the studied system in theory of optimal control.

We introduce the change of variables:

$$z(t,\mu) = \tilde{z}(t,\mu) + Hx(t,\mu), \tag{1.1.8}$$

$$x(t,\mu) = \tilde{x}(t,\mu) - \mu N\tilde{z}(t,\mu), \tag{1.1.9}$$

where the matrices $H = H(t,\mu) \leftrightarrow N = N(t,\mu)$ have dimensions respectively $m \times n,\ n \times m$ and will be determined by the parameters of the system (1.1.1). Later they called a matrix integral manifold.

From (1.1.8) and (1.1.9) we have the relation:

$$\begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} E_n - \mu NH & \mu N \\ -H & E_m \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \tag{1.1.10}$$

$$\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} E_n & -\mu N \\ H & E_m - \mu HN \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix}. \tag{1.1.11}$$

We denote

$$M = M(t,\mu) = \begin{pmatrix} E_n & -\mu N \\ H & E_m - \mu HN \end{pmatrix}, \tag{1.1.12}$$

then

$$M^{-1} = \begin{pmatrix} E_n - \mu NH & \mu N \\ -H & E_m \end{pmatrix}. \tag{1.1.13}$$

It is easy to verify that for known H and N, $M \cdot M^{-1} = E_{n+m}$.

Since $y = \begin{pmatrix} x \\ z \end{pmatrix}$, $\tilde{y} = \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix}$, the ratio (1.1.10) and (1.1.11) are written as

$$\tilde{y} = M^{-1} \cdot y, \qquad y = M \cdot \tilde{y}\,. \tag{1.1.14}$$

For non-stationary matrices H and N depend on t and μ. Then the second relation (1.1.14) we have

$$\dot{y} = \dot{M} \cdot \tilde{y} + M \cdot \dot{\tilde{y}}\,. \tag{1.1.15}$$

In view of (1.1.14) and (1.1.15), the system (1.1.1) is transformed as

$$\dot{\tilde{y}} = (M^{-1}AM - M^{-1}\dot{M})\tilde{y} + M^{-1}Bu + M^{-1}f \ . \qquad (1.1.16)$$

We write the equation (1.1.16) in expanded form

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \end{pmatrix} = \begin{pmatrix} \tilde{A}_1(t,\mu) + N(-\mu H\tilde{A}_1(t,\mu) + A_3 + A_4 H - \mu\dot{H}) & \mu\dot{N} - \mu\tilde{A}_1(t,\mu)N + N\tilde{A}_4(t,\mu) + A_2 + \mu N(\mu\dot{H} + \mu H\tilde{A}_1(t,\mu) - A_3 - A_4 H)N \\ \frac{1}{\mu}(-\mu\dot{H} - \mu\tilde{A}_1(t,\mu) + A_3 + A_4 H) & \frac{1}{\mu}\tilde{A}_4(t,\mu) + (\mu\dot{H} + \mu H\tilde{A}_1(t,\mu) - A_3 - A_4 H)N \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} +$$

$$+ \begin{pmatrix} \tilde{B}_1(t,\mu) \\ \frac{1}{\mu}\tilde{B}_2(t,\mu) \end{pmatrix} u + \begin{pmatrix} \tilde{f}_1(t,\mu) \\ \frac{1}{\mu}\tilde{f}_2(t,\mu) \end{pmatrix}, \qquad (1.1.17)$$

where

$$\tilde{A}_1(t,\mu) = A_1(t) + A_2(t)H(t,\mu), \qquad \tilde{A}_4(t,\mu) = A_4(t) - \mu H(t,\mu)A_2(t),$$

$$\tilde{B}_1(t,\mu) = B_1(t) + N(t,\mu)\tilde{B}_2(t,\mu), \qquad \tilde{B}_2(t,\mu) = B_2(t) - \mu H(t,\mu)B_1(t),$$

$$\tilde{f}_1(t,\mu) = f_1(t) + N(t,\mu)\tilde{f}_2(t,\mu), \quad \tilde{f}_2(t,\mu) = f_2(t) - \mu H(t,\mu)f_1(t), \quad (1.1.18)$$

$$u = u(t,\mu) .$$

In order to slow and fast state variables of the system (1.1.1) divided, we require the following conditions:

$$-\mu\dot{H} - \mu H\tilde{A}_1(t,\mu) + A_3 + A_4 H = 0 \ \text{ and}$$

$$\mu\dot{N} - \mu\tilde{A}_1(t,\mu)N + N\tilde{A}_4(t,\mu) + A_2 = 0 \ . \qquad (1.1.19)$$

Hence we have the following matrix equations from which we can determine the matrices-functions H and N:

$$\mu\dot{H} = -\mu HA_0 + A_3 + A_4 H - \mu HA_2\left(H + A_4^{-1}A_3\right), \qquad (1.1.20)$$

$$\mu\dot{N} = \mu A_0 N - NA_4 + \mu NHA_2 + \mu A_2\left(H + A_4^{-1}A_3\right)N - A_2\,, \qquad (1.1.21)$$

where $A_0 = A_1 - A_2 A_4^{-1} A_3$.

If H and N satisfy the equations (1.1.20) and (1.1.21) then from (1.1.17) we have

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \end{pmatrix} = \begin{pmatrix} \tilde{A}_1(t,\mu) & 0 \\ 0 & \dfrac{1}{\mu}\tilde{A}_4(t,\mu) \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} + \begin{pmatrix} \tilde{B}_1(t,\mu) \\ \dfrac{1}{\mu}\tilde{B}_2(t,\mu) \end{pmatrix} u + \begin{pmatrix} \tilde{f}_1(t,\varpi) \\ \dfrac{1}{\mu}\tilde{f}_2(t,\mu) \end{pmatrix}$$

and

$$\dot{\tilde{x}} = \tilde{A}_1(t,\mu)\tilde{x} + \tilde{B}_1(t,\mu)u + \tilde{f}_1(t,\mu)\,, \qquad (1.1.22)$$

$$\mu\dot{\tilde{z}} = \tilde{A}_4(t,\mu)\tilde{z} + \tilde{B}_2(t,\mu)u + \tilde{f}_2(t,\mu)\,. \qquad (1.1.23)$$

The boundary conditions for equations (1.1.22) and (1.1.23) are defined as

$$\tilde{x}\left(t_0\right) = \tilde{x}^0, \quad \tilde{x}\left(t_1\right) = \tilde{x}^1, \quad \tilde{z}\left(t_0\right) = \tilde{z}^0, \quad \tilde{z}\left(t_1\right) = \tilde{z}^1, \qquad (1.1.24)$$

where $\tilde{x}^i = x^i + \mu N(t_i)\tilde{z}^i$, $\tilde{z}^i = z^i - H(t_i)x^i$, $i = 0, \quad 1$.

Equation (1.1.22), (1.1.23) and the boundary conditions (1.1.24) can be rewritten as

$$\dot{\tilde{y}} = \tilde{A}(t,\mu)\tilde{y} + \tilde{B}(t,\mu)u + \tilde{f}(t,\mu)\,, \quad \tilde{y}(t_i) = \tilde{y}^i, \qquad i = 0,1, \qquad (1.1.25)$$

where

$$\tilde{y} = M^{-1}y = \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix}, \quad y = \begin{pmatrix} x \\ z \end{pmatrix}, \quad \tilde{B}(t,\mu) = M^{-1}B = \begin{pmatrix} \tilde{B}_1(t,\mu) \\ \dfrac{1}{\mu}\tilde{B}_2(t,\mu) \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \dfrac{1}{\mu}B_2 \end{pmatrix}, \qquad (1.1.26)$$

$\tilde{f}(t,\mu) = M^{-1} f = \begin{pmatrix} \tilde{f}_1(t,\mu) \\ \frac{1}{\mu}\tilde{f}_2(t,\mu) \end{pmatrix}$, $\tilde{x}$ and $\tilde{z}$ are determined from (1.1.8) and (1.1.9), and M^{-1} of (1.1.13),

$$\tilde{A}(t,\mu) = \begin{pmatrix} \tilde{A}_1(t,\mu) & 0 \\ 0 & \frac{\tilde{A}_4(t,\mu)}{\mu} \end{pmatrix}, \tag{1.1.27}$$

where $\tilde{A}_1(t,\mu)$ and $\tilde{A}_4(t,\mu)$ are determined by the relations of (1.1.18).

As a result, we have the following result as a theorem.

Theorem 1.1.1. Suppose that the conditions I, II and differentiable matrices functions $H(t,\mu)$, $N(t,\mu)$ are satisfying the equation (1.1.20) and (1.1.21). Then the system (1.1.1) may be divided into two subsystems of lower order, respectively, which contain slow and fast coordinate system, where they are connected only in the control.

Thus, if the conditions of theorem 1.1.1, we obtain a new system with traffic separation, which is equivalent to the original system, as it has all the properties (controllability and stabilizability) of the original system (1.1.1).

Therefore, in the study problems of control as constraints can take differential constraints (1.1.22), (1.1.23).

1.2 Matrix of Integral Manifolds

Here we study the equation (1.1.20), (1.1.21) from which are determined by the matrices of integral manifold. We prove the existence and uniqueness of solutions of the equation (1.1.20). Show more mutually conjugate two equations that correspond to linear homogeneous parts of (1.1.20) and (1.1.21).

Theorem 1.2.1. If $\Phi_*(t_1 t_0)$ is transition matrix for the equation $\dot{p}(t) = -A_0'(t)p(t)$, and $\Psi(t_1 t_0, \mu)$ for the equation $\mu\dot{z}(t,\mu) = A_4(t)z(t,\mu)$, equation (1.1.20) with the initial condition $H(t_0,\mu) = H_0, \quad (H_0 \in G, G \subset R^{m\times n}$ - bounded set) at $\mu > 0$ equivalent to the integral equation.

$$H(t,\mu) = \Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0) + \frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)A_3(s)\Phi_*'(t,s)ds -$$

$$-\int_{t_0}^{t}\Psi(t,s,\mu)H(s,\mu)A_2(s)\left(H(s,\mu) + A_4^{-1}(s)A_3(s)\right)\Phi_*'(\text{t,s})\text{ds} = \Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0) +$$

$$+\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\left(A_3(s) - \mu H(s,\mu)A_2(s)\left(H(s,\mu) + A_4^{-1}(s)A_3(s)\right)\right)\Phi_*'(\text{t,s})\text{ds}, \qquad (1.2.1)$$

$t \in [t_0, t_1]$.

Proof. Differentiating (1.2.1) with respect to t and using the equation:

$$\dot{\Phi}_*(t,t_0) = -A_0'(t)\Phi_*(t,t_0), \qquad (1.2.2)$$

$$\mu\dot{\Psi}(t,t_0,\mu) = A_4(t)\Psi(t,t_0,\mu), \qquad (1.2.3)$$

We obtain

$$\dot{H}(t,\mu) = \frac{1}{\mu}A_4(t)\Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0) - \Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0)A_0(t) +$$

$$+\frac{1}{\mu^2}A_4(t)\int_{t_0}^{t}\Psi(t,s,\mu)\left(A_3(s) - \mu H(s,\mu)A_2(s)\left(H(s,\mu) + A_4^{-1}(s)A_3(s)\right)\right)\Phi_*'(t,s)ds -$$

$$-\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\left(A_3(s) - \mu H(s,\mu)A_2(s)\left(H(s,\mu) + A_4^{-1}(s)A_3(s)\right)\right)\Phi_*'(t,s)dsA_0(t) +$$

$$+\frac{1}{\mu}A_3(t) - H(t,\mu)A_2(t)\left(H(t,\mu) + A_4^{-1}(t)A_3(t)\right) = \frac{1}{\mu}A_4(t)[\Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0) +$$

$$+\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\Big(A_3(s)-\mu H(s,\mu)A_2(s)\Big(H(s,\mu)+A_4^{-1}(s)A_3(s)\Big)\Big)\Phi_*'(t,s)ds]-$$

$$-[\Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0)+\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\Big(A_3(s)-\mu H(s,\mu)A_2(s)\Big(H(s,\mu)+A_4^{-1}(s)A_3(s)\Big)\Big)\Phi_*'(t,s)ds]A_0(t)+$$

$$+\frac{1}{\mu}A_3(t)+H(t,\mu)A_2(t)\Big(H(t,\mu)+A_4^{-1}(t)A_3(t)\Big)=\frac{1}{\mu}A_4(t)H(t,\mu)-H(t,\mu)A_0(t)+$$

$$+\frac{1}{\mu}A_3(t)-H(t,\mu)A_2(t)\Big(H(t,\mu)+A_4^{-1}(t)A_3(t)\Big)..$$

Hence we have the equation (1.1.20), Q.E.D.

We now show that (1.1.20) (or (1.2.1)) has a unique solution when $0<\mu\le\mu_0<1$, where μ_0 - a positive constant. We introduce the notation:

$$H_0(t,\mu)=\Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0)+\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)A_3(s)\Phi_*'(t,s)ds\,, \quad (1.2.4)$$

$$K(H,t,\mu)=-\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)H(s,\mu)A_2(s)\Big(H(s,\mu)+A_4^{-1}(s)A_3(s)\Big)\Phi_*'(t,s)ds\,. \quad (1.2.5)$$

Then the integral equation (1.2.1) is written as an operator equation

$$H(t,\mu)=H_0(t,\mu)+\mu K(H,t,\mu)\,, \quad (1.2.6)$$

where $K(H,t,\mu)$ - integral operator in the form (1.2.5).

At $\mu>0,\quad H_0\in G,\quad t\in[t_0,t_1]$ introduce the following notation:

$$M_1=\|H_0\|,\ M_2=\max_{t_0\le t\le t_1}\|A_2(t)\|,\ M_3=\max_{t_0\le t\le t_1}\|A_3(t)\|,$$

$$M_4=\max_{t_0\le t\le t_1}\|A_4^{-1}(t)\|\ M=\{M_1,\quad M_2,\quad M_3\quad M_4\}\,, \quad (1.2.7)$$

because by condition I and II of the matrices $A_2(t),\quad A_3(t)$ are uniformly bounded and $A_4(t)$ - stable matrix with $t\in[t_0,t_1]$, then we have

$$\left\|\Psi(t,s,\mu)\right\| \le C_1 e^{-\frac{\lambda_1(t-s)}{\mu}}, \quad \left\|\Phi'_*(\text{t,s})\right\| \le C_2 e^{\lambda_2(t-s)}, \tag{1.2.8}$$

$$\left(0 \le s \le t \le t_1, \quad 0 < \mu < \mu_0, \quad \lambda_1 > 0, \quad \lambda_2 > 0\right).$$

$$\left\|H_0(t,\mu)\right\| \le CMe^{-\frac{\lambda(t-t_0)}{\mu}} + \frac{MC}{\lambda}\left(1 - e^{-\frac{\lambda(t-t_0)}{\mu}}\right) \le \left(1 - \frac{1}{\lambda}\right)MCe^{-\frac{\lambda(t-t_0)}{\mu}} + \frac{MC}{\lambda} \le m_1, \tag{1.2.9}$$

where $m_1 > 0, \quad \lambda = \lambda_1\left(1 - \dfrac{\mu\lambda_2}{\lambda_1}\right) > 0, \; C = C_1 \cdot C_2, \quad \mu < \mu_1, \quad \mu_1 = \dfrac{\lambda_1}{\lambda_2}$.

At

$$\left\|H(t,\mu)\right\| \le m_2, \tag{1.2.10}$$

$$\left\|K(H,t,\mu)\right\| \le m_2\left(1 - e^{-\frac{\lambda(t-t_0)}{\mu}}\right)\left(m_2 + M^2\right) \le M^*. \tag{1.2.11}$$

Choose a number μ_2 such, that for any $\mu \le \mu_2$ there were

$$\left\|H_0(t,\mu) + \mu K(H,t,\mu)\right\| \le m_2.$$

For this it suffices choose

$$\mu_2 \le \min\left\{\mu_1, \frac{m_2 - m_1}{M^*}\right\}. \tag{1.2.12}$$

We construct a successive approximation $H_0, H_1, \ldots, H_k, \ldots$ by the formula

$$H_{k+1}(t,\mu) = H_0(t,\mu) + \mu K(H_k,t,\mu), \quad k = 0,1,2,\ldots \tag{1.2.13}$$

If $\left\|H_k(t,\mu)\right\| \le m_2$ at $t \in [t_0,t_1]$, then $H_{k+1}(t,\mu)$ is a continuous matrix function defined on $\left[t_0,t_1\right]$ and satisfying

$$\left\|H_{k+1}(t,\mu)\right\| \le \left\|H_0(t,\mu)\right\| + \mu\left\|K(H_k,t,\mu)\right\| \le m_1 + \mu M^* \le m_2. \tag{1.2.14}$$

For k=0 and k=1 have the inequality (1.2.14). By induction it holds for all $k \geq 0$.

Let the matrix $A_i(t), \quad (i=\overline{1,4})$ of the system (1.1.1) are defined, uniformly bounded and uniformly continuous together with its derivatives on $[t_0, t_1]$. As a $K(H,t,\mu)$ continuous function of its arguments, one can show that there exists a positive number L for any matrices functions $\overline{H}$ and $\overline{\overline{H}}$, satisfying the inequalities

$\left\|\overline{H}(t,\mu)\right\| \leq m_2$, $\left\|\overline{\overline{H}}(t,\mu)\right\| \leq m_2$, such that

$$\left\|K(\overline{H},t,\mu)-K(\overline{\overline{H}},t,\mu)\right\| \leq L\left\|\overline{H}-\overline{\overline{H}}\right\| \qquad (1.2.15)$$

when $\mu \leq \mu_2$. Subtract term by term from (1.2.13) with k=n-1 is the same equation for k=n-2. Then we obtain

$$H_n(t,\mu)-H_{n-1}(t,\mu)=\mu[K(H_{n-1},t,\mu)-K(H_{n-2},t,\mu)], \qquad (1.2.16)$$

in view of (1.2.15) from (1.2.16), we have

$$\left\|H_n(t,\mu)-H_{n-1}(t,\mu)\right\| \leq \mu L \max_{t_0 \leq t \leq t_1}\left\|H_{n-1}(t)-H_{n-2}(t)\right\|. \qquad (1.2.17)$$

We introduce the notation

$$r_n = \max_{t_0 \leq t \leq t_1}\left\|H_n(t,\mu)-H_{n-1}(t,\mu)\right\|, \qquad (1.2.18)$$

then from (1.2.17) we have

$$r_n \leq \mu L r_{n-1}. \qquad (1.2.19)$$

From the recurrence relation (1.2.17) we obtain

$$r_n \leq \mu L^{n-1} r_1, \qquad (1.2.20)$$

where $r_1 = \max\limits_{t_0 \le t \le t_1} \left\| H_1(t,\mu) - H_0(t,\mu) \right\| \le \mu \max\limits_{\substack{t_0 \le t \le t_1 \\ \|H\| \le m_2, \quad \mu < \mu_2}} \left\| K(H,t,\mu) \right\| = \mu M^*$.

From (1.2.20) follows that the above series converges uniformly for $t \in [t_0, t_1]$ any choice $\mu < \mu_3$, and where $\mu_3 L = 1$.

Now we will put $\mu^* = \min\{\mu_2, \mu_3\}$, then at $\mu < \mu^*$ creation of successive approximations (1.2.13) it is possible to, and the sequence $\{H_k(t,\mu)\}, \quad k = 0,1,\ldots$ converges uniformly on the interval $[t_0, t_1]$. Limit of this sequence satisfies (1.2.6) (or (1.2.1)).

The uniqueness of the solution follows from Gronwall's lemma, for any two solutions $\bar{H}$ и $\bar{\bar{H}}$ in the common domain of definition is valid assessment

$$\left\| \bar{H}(t,\mu) - \bar{\bar{H}}(t,\mu) \right\| \le \mu \left\| K(\bar{H},t,\mu) - K(\bar{\bar{H}},t,\mu) \right\| \le L \left\| \bar{H}(t,\mu) - \bar{\bar{H}}(t,\mu) \right\|.,$$

Is possible only if $\bar{H}(t,\mu) \equiv \bar{\bar{H}}(t,\mu)$.

Consider the matrix differential equations, which correspond to linear homogeneous parts of the equation (1.1.20), (1.1.21):

$$\mu \dot{\bar{H}} = -\mu \bar{H} A_0(t) + A_4(t) \bar{H} \ , \tag{1.2.21}$$

$$\mu \dot{\bar{N}} = \mu A_0(t) \bar{N} - \bar{N} A_4(t) . \tag{1.2.22}$$

where $\bar{H} = \bar{H}(t,\mu), \quad \bar{N} = \bar{N}(t,\mu), \quad t \in [t_0, t_1], \quad \bar{H} \in R^{m \times n}, \bar{N} \in R^{n \times m}$.

In the space $R^{m \times n}$ dot product [4]

$$\left(\bar{H}, \bar{N} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{h}_{ij} \bar{n}_{ji} = Sp(\bar{H} \cdot \bar{N}) . \tag{1.2.23}$$

We show that the equation (1.2.22) will be paired for a homogeneous matrix equation (1.2.21). Indeed, if the equation (1.2.22) is conjugate to (1.2.21), its solution $\bar{N}(t)$ for any t satisfies $\left(\bar{H}(t),\bar{N}(t)\right)=Sp\left(\bar{H}(t)\cdot\bar{N}(t)\right)=const$, where the $\bar{H}(t)$ - solution of the equation (1.2.21).

On the basis of this definition we have

$$0=\frac{d}{dt}Sp\left(\mu\bar{H}(t)\cdot\bar{N}(t)\right)=Sp\left(\mu\dot{\bar{H}}(t)\bar{N}(t)+\mu\bar{H}(t)\dot{\bar{N}}(t)\right)=$$
$$=Sp\left(-\mu\bar{H}(t)A_0(t)\cdot\bar{N}(t)+A_4(t)\bar{H}(t)\bar{N}(t)+\mu\bar{H}(t)\dot{\bar{N}}(t)\right).$$

Given the properties of the trace of the matrix, we obtain

$$0=Sp\left\{\bar{H}(t)[\mu\dot{\bar{N}}(t)-\mu A_0(t)\bar{N}(t)+\bar{N}(t)A_4(t)]\right\}.$$

This condition must be executed under any $\bar{H}(t),\quad \bar{N}(t)$.

Then $\mu\dot{\bar{N}}(t)-\mu A_0(t)\bar{N}(t)+\bar{N}(t)A_4(t)=0$.

Hence we have the equation (1.2.22). Now we write the equation (1.1.21) in the form

$$\mu\dot{N}(t)=\mu A_0(t)N(t)-N(t)A_4(t)+$$
$$+\mu\left[A_2(t)\left(H(t)+A_4^{-1}(t)A_3(t)\right)N(t)+N(t)H(t)A_2(t)\right]-A_2(t). \quad (1.2.24)$$

Let $H(t),\quad (t_0\le t\le t_1)$ - solution of the equation (1.1.20). Then, based on the basic property of the adjoint equation (it is the solution of the original differential equation backward in time), the equation (1.2.24) can be written as the integral equation

$$N(t)=\Phi(t,t_1)N_1\Psi'_*(t,t_1,\mu)+\int_{t_1}^{t}\Phi(t,s)\left[A_2(s)\left(H(s)+A_4^{-1}(s)A_3(s)\right)N(s)+N(s)H(s)A_2(s)\right]\Psi'_*(t,s,\mu)ds-$$
$$-\frac{1}{\mu}\int_{t_1}^{t}\Phi(t,s)A_2(s)\Psi'_*(t,s,\mu)ds, \quad (1.2.25)$$

where $N_1 = N(t_1)$, $\Phi\,(t,t_1)$, $\Psi_*(t_0,t_1,\mu)$ are the transition matrix for the equation $\dot{x}(t) = A_0(t)x(t)$, $\mu\dot{g}(t,\mu) = -A_4'(t)g(t,)$.

The existence and uniqueness of solutions of the equation (1.2.25) can be proved similarly to the previous case.

It should be noted that the differential equation (1.2.22), the boundary condition is not specified in the initial moment of time t_0, but in the end of the transition process. This follows from the basic properties of the ad joint equation.

1.3 Matrix Transition of Singularly Perturbed System and Its Asymptotic Behavior

Consider the question of constructing a transition matrix of the system (1.1.1). Suppose that the conditions I and II. Then for sufficiently small values of the parameter, the transition matrix $Y(t,t_0,\mu)$ of the system

$$\dot{y}(t,\mu) = A(t,\mu)y(t,\mu)\,,\ \ y(t_0) = y^0\,, \tag{1.3.1}$$

corresponding to the system (1.1.1) can be determined as the solution of the matrix differential equation

$$\dot{Y}(t,t_0,\mu) = A(t,\mu)Y(t,t_0,\mu), \quad Y(t_0,t_0,\mu) = E_{n+m}\,. \tag{1.3.2}$$

Matrix $Y(t,t_0,\mu)$ divided into blocks

$$Y(t,t_0,\mu) = \begin{pmatrix} Y_1(t,t_0,\mu) & \mu Y_2(t,t_0,\mu) \\ Y_3(t,t_0,\mu) & \mu Y_4(t,t_0\mu) \end{pmatrix}, \tag{1.3.3}$$

in view of (1.3.3) from the (1.3.2) we have:

$$\dot{Y}_1 = A_1(t)Y_1 + A_2(t)Y_3, \quad Y_1(t_0,t_0,\mu) = E_n\,, \tag{1.3.4}$$

$$\mu\dot{Y}_3 = A_3(t)Y_1 + A_4(t)Y_3\,,\ \ Y_3(t_0,t_0,\mu) = 0\,,$$

$$\dot{Y}_2 = A_1(t)Y_2 + A_2(t)Y_4,\ Y_2(t_0, t_0, \mu) = 0, \tag{1.3.5}$$

$$\mu \dot{Y}_4 = A_3(t)Y_2 + A_4(t)Y_4,\ Y_4(t_0, t_0, \mu) = \frac{1}{\mu} E_m .$$

Solution of the Cauchy problem (1.3.4) and (1.3.5) will be determined in the form [17]:

$$Y_i(t, t_0, \mu) = \overline{Y}_i(t, t_0, \mu) + \Pi Y_i(\tau, \mu), \quad i = 1, 2, 3, 4, \tag{1.3.6}$$

where

$$\overline{Y}_i(t, t_0, \mu) = \sum_{k=0}^{\infty} Y_{ik}(t, t_0)\mu^k, \quad i = 1, 2, 3, 4, \tag{1.3.7}$$

$$\Pi Y_i(\tau, \mu) = \sum_{k=0}^{\infty} \Pi_k Y_i(\tau)\mu^k, i = 1, 2, 3,$$

$$\Pi Y_4(\tau, \mu) = \frac{1}{\mu}\Pi_{-1}Y_4(\tau) + \sum_{k=0}^{\infty} \Pi_k Y_4(\tau)\mu^k, \quad \tau = \frac{t - t_0}{\mu},$$

$$\Pi Y_4(\tau, \mu) = \frac{1}{\mu}\Pi_{-1}Y_4(\tau) + \sum_{k=0}^{\infty} \Pi_k Y_4(\tau)\mu^k, \quad \tau = \frac{t - t_0}{\mu}.$$

We substitute (1.3.6) to (1.3.4) and (1.3.5). Further in these systems after function replacement $\overline{\mathrm{Y}}_i \cdot (t, t_0, \mu),\ \Pi \mathrm{Y}_i(t, \mu)$ decomposition (1.3.7), equating the coefficients of like powers, and separately depending on t, and separately depending on the τ, we obtain the equation for determining the terms of the expansion (1.3.7).

At μ^{-1} with regard τ

$$\frac{d\Pi_{-1}Y_4(\tau)}{d\tau} = A_{40}(t_0)\Pi_{-1}Y_4(\tau),\ \Pi_{-1}Y_4(0) = E_m . \tag{1.3.8}$$

At μ^0 with regard t:

$$\frac{d\overline{Y}_{10}(t,t_0)}{dt}=(A_1-A_2A_4^{-1}A_3)\overline{Y}_{10}(t,t_0),\ \ Y_{10}(t_0,t_0)=E_n\,, \tag{1.3.9}$$

$$\overline{Y}_{30}(t,t_0)=-A_4^{-1}A_3\overline{Y}_{10}(t,t_0)\,,$$

$$\frac{d\overline{Y}_{20}(t,t_0)}{dt}=(A_1-A_2A_4^{-1}A_3)\overline{Y}_{20}(t,t_0)\,,\ \overline{Y}_{20}(t_0,t_0)=-A_{20}(t_0)A_{40}^{-1}(t_0)\,, \tag{1.3.10}$$

$$\overline{Y}_{40}(t\ \ ,t_0)=-A_4^{-1}A_3\overline{Y}_{20}(t,t_0)\,.$$

At μ^0 with regard τ :

$$\frac{d\Pi_0Y_1(\tau)}{d\tau}=0,\ \ \frac{d\Pi_0Y_3(\tau)}{d\tau}=A_{30}(t_0)\Pi_0Y_1(\tau)+A_{40}(t_0)\Pi_0Y_3(\tau)\,, \tag{1.3.11}$$

$$\frac{d\Pi_0Y_2(\tau)}{d\tau}=A_{20}(t_0)\Pi_{-1}Y_4(\tau), \tag{1.3.12}$$

$$\frac{d\Pi_0Y_4(\tau)}{d\tau}=A_{30}(t_0)\Pi_0Y_2(\tau)+A_{40}(t_0)\Pi_0Y_4(\tau)+A_{41}(t_0)\Pi_{-1}Y_4\,,$$

where $A_{i0}(t_0)=A_i(t_0), i=1,2,3,4$.

For the (1.3.8) - (1.3.12) we have the initial conditions:

$$\overline{Y}_{10}(t_0,t_0)+\Pi_0Y_1(0)=E_n,\ \ \overline{Y}_{30}(t_0,t_0)+\Pi_0Y_3(0)=0\,, \tag{1.3.13}$$

$$\overline{Y}_{20}(t_0,t_0)+\Pi_0Y_2(0)=0,\ \ \ \overline{Y}_{40}(t_0,t_0)+\Pi_0Y_4(0)=0\,.$$

The solution of equation (1.3.8) written in the form

$$\Pi_{-1}Y_4(\tau)=\exp\left(A_4(t_0)\tau\right). \tag{1.3.14}$$

Solutions of system (1.3.9) - (1.3.12) with the initial conditions (1.3.13) are given by:

$$\Pi_0Y_1(\tau)=0\,,\ \Pi_0Y_2(\tau)=A_{20}(t_0)A_{40}^{-1}(t_0)\exp\left(A_{40}(t_0)\tau\right), \tag{1.3.15}$$

$$\Pi_0Y_3(\tau)=\exp\left(A_{40}(t_0)\tau\right)A_4^{-1}(t_0)A_3(t_0)\,,$$

$$\Pi_0 Y_4(\tau) = -\exp\left(A_4(t_0)\tau\right) A_{40}^{-1}(t_0) A_{30}(t_0) A_{20}(t_0) A_{40}^{-1}(t_0) +$$

$$+\int_0^{\tau} \exp\left(A_4(t_0)(\tau - s)\right)\left(A_{41}(t_0) + A_{30}(t_0) A_{20}(t_0) A_{40}^{-1}(t_0)\right)\exp\left(A_{40}(t_0)s\right)ds,$$

$$\overline{Y}_{10}(t,t_0) = \exp\left(\int_{t_0}^{t}\left(A_1(\sigma) - A_2(\sigma)A_4^{-1}(\sigma)A_3(\sigma)\right)d\sigma\right), \qquad (1.3.16)$$

$$\overline{Y}_{30}(t\ ,t_0) = -A_4^{-1}(t_0)A_3(t_0)\overline{Y}_{10}(t,t_0)\ ,$$

$$\overline{Y}_{20}(t,t_0) = -\exp\left(\int_{t_0}^{t}\left(A_1(\sigma) - A_2(\sigma)A_4^{-1}(\sigma)A_3(\sigma)\right)d\sigma\right)\cdot A_{20}(t_0)A_{40}^{-1}(t_0)\ ,$$

$$\overline{Y}_{40}(t\ ,t_0) = -A_4^{-1}(t_0)A_3(t_0)\overline{Y}_{20}(t,t_0)\ .$$

For the matrix $\Pi_0 Y_i(\tau)$, i=2,3,4 the following estimated holds on the interval $(0,\infty)$ [18]:

$$\left\|\Pi_0 Y_i(\tau)\right\| \le C\cdot\exp(-\beta\tau), \quad (\tau \ge 0, \quad \beta > 0, \quad i = 2,3,4)\ . \qquad (1.3.17)$$

Proceeding similarly with the terms of the expansion of the order μ^p can be with respect to t write the following:

$$\frac{d\overline{Y}_{1p}}{dt} = A_1(t)\overline{Y}_{1p}(t,t_0) + A_2(t)\overline{Y}_{3p}(t,t_0), \qquad (1.3.18)$$

$$\frac{d\overline{Y}_{3p}}{dt} = A_3(t)\overline{Y}_{1p}(t,t_0) + A_4(t)\overline{Y}_{3p}(t,t_0)\ ,$$

$$\frac{d\overline{Y}_{2p}}{dt} = A_1(t)\overline{Y}_{2p}(t,t_0) + A_2(t)\overline{Y}_{4p}(t,t_0),$$

$\dfrac{d\overline{Y}_{4,p-1}}{dt} = A_3(t)\overline{Y}_{2p}(t,t_0) + A_4(t)\overline{Y}_{4p}(t,t_0)$, with regard τ :

$$\frac{d\Pi_p Y_1(\tau)}{d\tau} = d_{1p}(\tau), \quad \frac{d\Pi_p Y_3(\tau)}{d\tau} = A_{30}(t_0)\Pi_p Y_1(\tau) + A_{40}(t_0)\Pi_p Y_3(\tau) + d_{3p}(\tau), \tag{1.3.19}$$

$$\frac{d\Pi_p Y_2(\tau)}{d\tau} = d_{2p}(\tau), \quad \frac{d\Pi_p Y_4(\tau)}{d\tau} = A_{30}(t_0)\Pi_p Y_2(\tau) + A_{40}(t_0)\Pi_p Y_4(\tau) + d_{4p}(\tau),$$

where $d_{1p}(\tau) = \sum_{j=0}^{p-1} A_{1,p-1-j}(t_0)\Pi_j Y_1(\tau) + \sum_{j=0}^{p-1} A_{2,p-1-j}(t_0)\Pi_j Y_3(\tau)$,

$$d_{2p}(\tau) = A_{2p}(t_0)\Pi_{-1}Y_4(\tau) + \sum_{j=0}^{p-1} A_{1,p-1-j}(t_0)\Pi_j Y_2(\tau) + \sum_{j=0}^{p-1} A_{2,p-j}(t_0)\Pi_j Y_4(\tau),$$

$$d_{3p}(\tau) = \sum_{j=0}^{p-1} A_{3,p-j}(t_0)\Pi_j Y_1(\tau) + \sum_{j=0}^{p-1} A_{4,p-j}(t_0)\Pi_j Y_3(\tau),$$

$$d_{4p}(\tau) = A_{4p+1}(t_0)\Pi_{-1}Y_4(\tau) + \sum_{j=0}^{p-1} A_{3,p-j}(t_0)\Pi_j Y_2(\tau) + \sum_{j=0}^{p-1} A_{4,p-j}(t_0)\Pi_j Y_4(\tau).$$

The initial conditions for the equation (1.3.18), (1.3.19) is determined by the condition:

$$\overline{Y}_{1p}(t_0, t_0) + \Pi_p Y_1(0) = 0, \quad \overline{Y}_{2p}(t_0, t_0) + \Pi_p Y_2(0) = 0, \tag{1.3.20}$$

$$\overline{Y}_{3p}(t_0, t_0) + \Pi_p Y_3(0) = 0, \quad \overline{Y}_{4p}(t_0, t_0) + \Pi_p Y_4(0) = 0.$$

Suppose now that defined terms in the expansions (1.3.7) up to order p inclusive, i.e. obtained the following partial sums

$$Y_{ip}(t, t_0, \mu) = \sum_{j=0}^{p}\left[\overline{Y}_{ij}(t, t_0) + \Pi_j Y_i(\tau)\right]\mu^j, \quad i = 1, 2, 3, \tag{1.3.21}$$

$$Y_{4p}(t, t_0, \mu) = \frac{1}{\mu}\Pi_{-1}Y_4(\tau) + \sum_{j=0}^{p}\left[\overline{Y}_{4j}(t, t_0) + \Pi_j Y_4(\tau)\right]\mu^j.$$

Then solutions of systems of equations (1.3.4), (1.3.5) may be represented as

$$Y_i(t, t_0, \mu) = Y_{ip}(t, t_0, \mu) + \phi_i(t, \mu), \; i = 1, 2, 3, 4. \tag{1.3.22}$$

Thus, if the conditions I and II, that for sufficiently small values of the parameter μ for matrix functions $\phi_i(t,\mu)$ and $\Pi_j Y_i(\tau)$ will we have the estimates:

$$\left\|\phi_i(t,\mu)\right\| \le C\mu^{p+1},\ \ i=1,2,3,4,\quad t\in[t_0,t_1], \tag{1.3.23}$$

$$\left\|\Pi_j Y_i(\tau)\right\| \le C\exp(-\beta\tau),\quad \tau\ge 0,$$

where C, β -const.

Now from the decomposition (1.3.7) select terms which form the zero-order approximation of the transition matrix $Y(t,t_0\mu)$. Zero approximation is denoted by $Y_0(t,t_0,\mu)$. Consider the degenerate system, which is obtained from (1.3.1) at $\mu=0$

$$\dot{\bar{x}} = A_1(t)\bar{x} + A_2(t)\bar{z},\ \ 0 = A_3(t)\bar{x} + A_4(t)\bar{z}. \tag{1.3.24}$$

If after $x(t,\mu)=\bar{x}(t,\mu)+\Pi x(\tau,\mu)$ and $z(t,\mu)=\bar{z}(t,\mu)+\Pi z(\tau,\mu)$ denote the solution of the Cauchy problem (1.3.1), and through $x^0(t)$ and $z^0(t)$ solution of the degenerate system (1.3.24) provided $x(t_0)=x^0$, then the conditions of theorem Tikhonov, we have

$$x(t,\mu)=x^0(t)+O(\mu),\ \ z(t,\mu)=z^0(t)+\frac{1}{\mu}\Pi_{-1}z(\tau)+O(\mu), \tag{1.3.25}$$

where $t\in\left[t_0,t_1\right]$, $\dfrac{1}{\mu}\Pi_{-1}z(\tau)$ - the first term of the expansion frontier function $\Pi z(\tau)$. Applying this provision to the problem (1.3.2), we can write the zero approximation $Y_0(t,t_0,\mu)$ transition matrix $Y(t,t_0\mu)$, which gives a uniform asymptotic accuracy $O(\mu)$ at all of interest to us time interval

$$Y_0(t,t_0,\mu)=\begin{pmatrix} \exp\left(\int\limits_{t_0}^{t} A_0(s)ds\right) & 0 \\ -A_4^{-1}(t_0)A_3(t_0)\exp\left(\int\limits_{t_0}^{t} A_0(s)ds\right)+\exp\left(A_4(t_0)\frac{t-t_0}{\mu}\right)A_4^{-1}(t_0)A_3(t_0) & \exp\left(A_4(t_0)\frac{t-t_0}{\mu}\right) \end{pmatrix}.$$

The continuity of the matrix $A_i(t)$, $i=1,2,3,4$ on the interval $[t_0,t_1]$ follows that at each point $t\in[t_0,t_1]$ derivative of the function $M(t)=\int\limits_{t_0}^{t} A_0(\lambda)d\lambda$ the upper limit is equal to the integrand function $A_0(t)$, i.e. $\dfrac{dM(t)}{dt}=A_0(t)$.

We have the following

Theorem 1.3.1. Matrix $Y_0(t,t_0,\mu)$ is the solution of the matrix differential equation

$$\dot{Y}_0(t,t_0,\mu)=\dot{A}_*(t)Y_0(t,t_0,\mu),\ \ Y_0(t_0,t_0)=E_{n+m}\,, \tag{1.3.26}$$

where $A_*(t,\mu)=\begin{pmatrix} A_0(t) & 0 \\ -A_4^{-1}(t_0)A_3(t_0)A_0(t)+\dfrac{1}{\mu}A_3(t_0) & \dfrac{1}{\mu}A_4(t_0) \end{pmatrix}.$

Proof. We represent the matrix $Y_0(t,t_0,\mu)$ in the form

$$Y_0(t,t_0,\mu)=V\overline{Y}_0(t,t_0,\mu)V^{-1}, \tag{1.3.27}$$

where $V=\begin{pmatrix} E_n & 0 \\ -A_4^{-1}(t_0)A_3(t_0) & E_m \end{pmatrix}$, $\overline{Y}_0(t,t_0,\mu)=\begin{pmatrix} e^{\int\limits_{t_0}^{t} A_0(\lambda)d\lambda} & 0 \\ 0 & e^{A_4(t_0)\left(\frac{t-t_0}{\mu}\right)} \end{pmatrix}.$

We introduced the notation $\overline{A}(t,\mu)=\begin{pmatrix} A_0(t) & 0 \\ 0 & \dfrac{A_4(t_0)}{\mu} \end{pmatrix}.$

Then differentiating both sides of (1.3.27), we obtain

$$\frac{dY(t,t_0,\mu)}{dt} = V\frac{d\overline{Y}_0(t,t_0,\mu)}{dt}V^{-1} = V\overline{A}(t,\mu)\overline{Y}_0(t,t_0,\mu)V^{-1} = V\overline{A}(t,\mu)V^{-1}V\overline{Y}_0(t,t_0,\mu)V^{-1} =$$
$$= V\overline{A}(t,\mu)V^{-1}Y_0(t,t_0,\mu) = A_*(t,\mu)Y_0(t,t_0,\mu)\,.$$

At $t = t_0$ $Y_0(t_0,t_0) = E_{n+m}$, Q.E.D.

Theorem 1.3.2. If the condition (1.1.2), then the matrix $A_*(t,\mu)$ defines a system for which there exists an integral manifold

$$\overline{z} = -A_4^{-1}(t_0)A_3(t_0)\overline{x} + \tilde{z}\,. \tag{1.3.28}$$

The movement, which is described by the system

$$\dot{\overline{x}}(t) = A_0(t)\overline{x}(t),\ \ \overline{x}(t_0) = x^0, \tag{1.3.29}$$
$$\mu\dot{\tilde{z}}(t) = A_4(t_0)\tilde{z}(t),\ \ \tilde{z}(t_0) = \tilde{z}^0,$$

where $\tilde{z}^0 = A_4^{-1}(t_0)A_3(t_0)x^0 + z^0$.

In this case, there is a limit relation $\lim\limits_{\mu\to 0}\tilde{z}(t,\mu) = 0$ or $\lim\limits_{\mu\to 0}\overline{z}(t,\mu) = -A_4^{-1}(t_0)A_3(t_0)\overline{x}^0(t) = \tilde{z}^0(t)$.

Proof.

Indeed, the system $\dot{\overline{y}}(t) = A_*(t,\mu)\overline{y}(t)$, $\overline{y}(t_0) = y^0$, where $y(t) = col(\overline{x}(t),\overline{z}(t))$ in expanded form is written as

$$\dot{\overline{x}}(t) = A_0(t)\overline{x}(t),\ \ \overline{x}(t_0) = x^0, \tag{1.3.30}$$
$$\dot{\overline{z}}(t) = -A_4^{-1}(t_0)A_3(t_0)A_0(t)\overline{x}(t) + \frac{1}{\mu}A_3(t_0)\overline{x}(t) + \frac{1}{\mu}A_4(t_0)\overline{z}(t),\ \ \overline{z}(t_0) = z^0$$

In view of (1.3.28) from the second equation of (1.3.30), we obtain

$\mu\dot{\tilde{z}}(t) = \mu\left(-A_4^{-1}(t_0)A_3(t_0)\dot{\bar{x}}(t) + \dot{\bar{z}}(t)\right) = -\mu A_4^{-1}(t_0)A_3(t_0)A_0(t)\bar{x}(t) + A_3(t_0)\bar{x}(t) - A_3(t_0)x(t) + A_4(t_0)\tilde{z}$

or $\mu\dot{\tilde{z}}(t) = A_4(t_0)\tilde{z}(t),\quad \tilde{z}(t_0) = \tilde{z}^0,$ where $\tilde{z}^0 = A_4^{-1}(t_0)A_3(t_0)x^0 + z^0$. This, together with the first equation (1.3.30) gives the system (1.3.29) and its solution can be written as:

$$\bar{x}^0(t) = \exp(\int_{t_0}^{t} A_0(\lambda)dx)x^0,\ \ \tilde{z}^0(t,\mu) = \exp(A_4(t_0)(t-t_0)/\mu)\tilde{z}^0 \text{ or}$$

$$\bar{z}^0(t,\mu) = -A_4^{-1}(t_0)A_3(t_0)\bar{x}^0(t) + \exp(A_4(t_0)(t-t_0)/\mu).$$

By hypothesis, the eigenvalues $\lambda_i(t_0)$ matrix $A_4(t_0)$ satisfy the inequality $\operatorname{Re}\lambda_i(t_0) < -\gamma < 0$ and $\mu \to 0$ we get the specified limit relations. At $\mu << 1$ will have the representation $\bar{z}^0(t,\mu) = \bar{z}^0(t) + O(e^{-\gamma\frac{(t-t_0)}{\mu}}),\quad \gamma > 0$.

1.4 Converting Matrix Transition on the Integral Manifold

We now state and prove a theorem which gives a formula of the transition matrix of the system (1.2.1) and allows you to split the state vector of the system to slow and fast components.

Theorem 1.4.1. Let the matrices $\Phi(t,s,\mu)$ and $\Psi(t,s,\mu)$ are transition matrices of homogeneous systems $\dot{\tilde{x}} = \tilde{A}_1\tilde{x}$, $\mu\cdot\dot{\tilde{z}} = \tilde{A}_4\cdot\tilde{z}$ i.e. they satisfy the equations

$$\dot{\Phi}(t,s,\mu) = \tilde{A}_1(t,\mu)\Phi(t,s,\mu),\ \ \Phi(s,s,\mu) = E_n\,, \tag{1.4.1}$$

$$\mu\dot{\Psi}(t,s,\mu) = \tilde{A}_4(t,\mu)\Psi(t,s,\mu)\,,\ \ \Psi(s,s,\mu) = E_m/\mu\,, \tag{1.4.2}$$

where matrices $\tilde{A}_1(t,\mu)$ and $\tilde{A}_4(t,\mu)$ are determined from (1.1.18). Then the transition matrix $Y(t,s,\mu)$ systems (1.3.1), the corresponding system (1.1.1), can be represented as

$$Y(t,s,\mu) = M(t,\mu)G(t,s,\mu)M^{-1}(s,\mu)\,, \tag{1.4.3}$$

where matrices $M(t,\mu)$ and $M^{-1}(t,\mu)$ are determined from (1.1.12) and (1.1.13), respectively;

$$G(t,s,\mu) = diag\left(\Phi(t,s,\mu), \quad \Psi(t,s,\mu)\right), \tag{1.4.4}$$

Matrix $\Phi(t,s,\mu)$ and $\Psi(t,s,\mu)$ will be called the transition matrix of slow and fast subsystems (1.3.1).

Proof. Matrix $Y(t,s,\mu)$ we will divide into blocks in the form of

$$Y\left(t,s,\mu\right) = \begin{pmatrix} Y_1\left(t,s,\mu\right) & \mu Y\left(t,s,\mu\right) \\ Y_3\left(t,s,\mu\right) & \mu Y\left(t,s,\mu\right) \end{pmatrix}. \tag{1.4.5}$$

Then from the equation

$$\dot{Y}\left(t,s,\mu\right) = A\left(t,\mu\right)Y\left(t,s,\mu\right),\; Y\left(t,s,\mu\right) = E_{m+m} \tag{1.4.6}$$

we have

$$\dot{Y}_1 = A_1\left(t\right)Y_1 + A_2\left(t\right)Y_3,\; Y\left(s,s,\mu\right) = E_m,\; \mu\dot{Y}_3 = A_3\left(t\right)Y_1 + A_4\left(t\right)Y_3,$$
$$Y_3\left(s,s,\mu\right) = 0 \tag{1.4.7}$$

$$\dot{Y}_2 = A_1\left(t\right)Y_2 + A_2\left(t\right)Y_4,\; Y_2\left(s,s,\mu\right) = E_m,\; \mu\dot{Y}_4 = A_3\left(t\right)Y_2 + A_4\left(t\right)Y_4,$$
$$Y_4\left(s,s,\mu\right) = \frac{1}{\mu}E_m \tag{1.4.8}$$

Now, in these systems, we make the change of variables

$$Y_3 = H(t,\mu)Y_1 + Z\,, \tag{1.4.9}$$

$$Y_4 = H(t,\mu)Y_2 + \Psi\,, \tag{1.4.10}$$

where $Y_1 = Y_1(t,s,\mu)$, $Y_2 = Y_2(t,s,\mu)$, $Y_3 = Y_3(t,s,\mu)$, $Y_4 = Y_4(t,s,\mu)$, $\Psi = \Psi(t,s,\mu)$.

$H(t,\mu)$, $Z = Z(t,s,\mu)$ - matrices of order $m \times n$, elements which are regularly depend on μ. Then the system (1.4.7) takes the form:

$$\dot{Y}_1 = A_1(t)Y_1 + A_2(t)H(t,\mu)Y_1 + A_2(t)Z = (A_1(t) + A_2(t)H(t,\mu))Y_1 + A_2(t)Z \ ,$$

$$\dot{Y}_1 = \tilde{A}_1(t,\mu)Y_1 + A_2(t)Z \ , \ Y_1(s,s,\mu) = E_n \ , \tag{1.4.11}$$

$$\mu(\dot{H}(t,\mu)Y_1 + H(t,\mu)\dot{Y}_1 + \dot{Z}) = A_3(t)Y_1 + A_4(t)H(t,\mu)Y_1 + A_4(t)Z,$$

$$(\mu\dot{H}(t,\mu) + \mu H(t,\mu)\tilde{A}_1(t,\mu))Y_1 + \mu H(t,\mu)A_2(t)Z + \mu\dot{Z} = (A_3(t) + A_4(t)H(t,\mu))Y_1 + A_4(t)Z \ ,$$

$$(\mu\dot{H}(t,\mu) + \mu H(t,\mu)\tilde{A}_1(t,\mu))Y_1 + \mu\dot{Z} = (A_3(t) + A_4(t)H(t,\mu))Y_1 + (A_4(t) - \mu H(t,\mu)A_2(t))Z.$$

In view of (1.1.19), we obtain from the last equation

$$\mu\dot{Z} = \tilde{A}_4(t,\mu)Z, \ \ Z(s,s,\mu) = -H(s,\mu) \, . \tag{1.4.12}$$

Substituting (1.4.10) into the first equation (1.4.8), we obtain

$$\dot{Y}_2 = \tilde{A}_1(t,\mu)Y_2 + A_2(t)\Psi, \ \ Y_2(s,s,\mu) = 0 \, . \tag{1.4.13}$$

Differentiating function (1.4.10) over the t obtain $\dot{Y}_4 = \dot{H}(t,\mu)Y_2 + H(t,\mu)\dot{Y}_2 + \dot{\Psi}$. The meaning Y_4 substitute into the second equation (1.4.8)

$$\mu\dot{H}(t,\mu)Y_2 + \mu H(t,\mu)\dot{Y}_2 + \mu\dot{\Psi} = A_3(t)Y_2 + A_4(t)H(t,\mu)Y_2 + A_4(t)\Psi,$$

$$\begin{gathered}\mu\dot{H}(t,\mu)Y_2 + \mu H(t,\mu)\tilde{A}_1(t,\mu)Y_2 + \mu H(t,\mu)A_2(t)\Psi + \mu\dot{\Psi} = \\ (A_3(t) + A_4(t)H(t,\mu))Y_2 + A_4(t)\Psi \, .\end{gathered}$$

Given the ratio from (1.1.19) as a result we have an equation with respect Ψ

$$\mu\dot{\Psi} = \tilde{A}_4(t,\mu)\Psi, \quad \Psi(s,s,\mu) = \frac{1}{\mu}E_m \, . \tag{1.4.14}$$

Now we eliminate Z from the equation (1.4.11). For this we introduce the matrix

$F = F(t,s,\mu)$ considering that the matrix $H(t,\mu)$ already known

$$F = Y_1 + \mu N(t,\mu)Z \; . \tag{1.4.15}$$

Then the systems (1.4.11) and (1.4.12) take the form:

$$\dot{F} = \tilde{A}_1(t,\mu)F \; , \; F(s,s,\mu) = E_n - \mu N(s,\mu)H(s,\mu) \, , \tag{1.4.16}$$

$$\mu\dot{Z} = \tilde{A}_4(t,\mu)Z \, , \; Z(s,s,\mu) = -H(s,\mu) \, , \tag{1.4.17}$$

where matrix $N(t,\mu)$ is determined from the second equation (1.1.19).

We now show that the matrices F, Z and Y_2 are defined in terms of matrices $\Phi(t,s,\mu)$ and $\Psi(t,s,\mu)$. By hypothesis 1.4.1 the matrices Φ and Ψ satisfy equations (1.4.1) and (1.4.2), then the matrices:

$$F = \Phi(E_n - \mu N(s,\mu)H(s,\mu)) \, , \tag{1.4.18}$$

$$Z = -\Psi H(s,\mu) \, , \tag{1.4.19}$$

$$Y_2 = \int_s^t \Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu)d\sigma \, , \tag{1.4.20}$$

satisfy the equations (1.4.16), (1.4.17) and (1.4.13), respectively. This statement for the first two relations is easily verified. We present the following lemma.

Lemma. Let the matrix $N(t,\mu)$ $\left(t \in [t_0, t_1], \quad \mu > 0\right)$ is the solution of the second equation in (1.1.19). Then

$$\frac{1}{\mu}\int_s^t \Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu)d\sigma = \Phi(t,s,\mu)N(s,\mu) - N(t,\mu)\Psi(t,s,\mu) \, , \tag{1.4.21}$$

Proof. Equation (1.4.21) can be written in the form

$$\frac{1}{\mu}\int_s^t \Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu)d\sigma = -\Phi(t,\sigma,\mu)N(\sigma,\mu)\Psi(\sigma,s,\mu)\Big|_s^t \tag{1.4.22}$$

Then from (1.4.22) obtain

$$\frac{d\left(\Phi(t,\sigma,\mu)N(\sigma,\mu)\Psi(\sigma,s,\mu)\right)}{d\sigma}=-\frac{1}{\mu}\Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu).$$

Using one of the properties of the transition matrix, hence we have

$$-\tilde{A}_1(\sigma,\mu)\Phi(t,\sigma,\mu)N(\sigma,\mu)\Psi(\sigma,s,\mu)+\Phi(t,\sigma,\mu)\frac{d\left(N(\sigma,\mu)\right)}{d\sigma}\Psi(\sigma,s,\mu)+$$
$$+\frac{1}{\mu}\Phi(t,\sigma,\mu)N(\sigma,\mu)\tilde{A}_4(\sigma,\mu)\Psi(\sigma,s,\mu)=-\frac{1}{\mu}\Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu),$$

$$-\tilde{A}(t,\mu)N(t,\mu)\Psi(t,s,\mu)+\dot{N}(t,\mu)\Psi(t,s,\mu)$$
$$+\frac{1}{\mu}\tilde{A}_4(t,\mu)\Psi(t,s,\mu)+\frac{1}{\mu}A_2(t)\Psi(t,s,\mu)=0$$

Multiplying this equality on the right by the matrix $\Phi(s,t,\mu)$ have

$$\mu\dot{N}(t,\mu)-\mu\tilde{A}_1(t,\mu)N(t,\mu)=-A_2(t)-N(t,\mu)\tilde{A}_4(t,\mu)$$

As a result, we obtain the second equation in (1.1.19), since by hypothesis matrix lemma $N(t,\mu)$ is the solution of the second equation (1.1.19). Therefore, the formula (1.4.21) is true. Q.E.D.

On the basis of the formula (1.4.21), the expression (1.4.20) can be written as

$$Y_2=\mu\left(\mathit{\Phi}N(s,\mu)-N(t,\mu)\Psi\right). \qquad (1.4.23)$$

Differentiating function (1.4.21) to the t and considering that $N(t,\mu)$ is the solution of the second equation (1.1.19), we obtain

$$\dot{Y}_2=\mu\tilde{A}_1(t,\mu)\Phi N(s,\mu)+N(t,\mu)\tilde{A}_4(t,\mu)\Psi-\mu\tilde{A}_1(t,\mu)N(t,\mu)\Psi$$
$$+A_2(t)\Psi-N(t,\mu)\tilde{A}_4(t,\mu)\Psi=$$

$$=\mu\tilde{A}_1(t,\mu)(\Phi N(s,\mu)-N(t,\mu)\Psi)+A_2(t)\Psi=\tilde{A}_1(t,\mu)Y_2+A_2(t)\Psi.$$

At $t = s$ rom the (1.4.20) it follows that $Y_2(s,s,\mu) = 0$.

From the relations (1.4.9), (1.4.10), (1.4.15), (1.4.18), (1.4.19), (1.4.21) we have: $Y_1 = \Phi(E_n - \mu N(s,\mu)H(s,\mu)) + \mu N(t,\mu)\Psi H(s,\mu)$,

$$Y_2 = \Phi N(s,\mu) - N(t,\mu)\Psi,$$

$$Y_3 = H(t,\mu)\Phi(E_n - \mu N(s,\mu)H(s,\mu)) - (E_m - \mu H(t,\mu)N(t,\mu))\Psi H(s,\mu), \quad (1.4.24)$$

$$Y_4 = H(t,\mu)\Phi N(s,\mu) + \frac{1}{\mu}(E_m - \mu H(t,\mu)N(s,\mu))\Psi.$$

Substituting the values of Y_i (i=1,2,3,4) from (1.4.24) the right-hand side of (1.4.5) we obtain (1.4.3). Q.E.D.

It should be noted that the system (1.2.1) may be replaced by an integral equation (Cauchy formula):

$$y(t,\mu) = Y(t,s,\mu)y\left(s,\mu\right) + \int_s^t Y(t,s,\mu)B(s,\mu)u(s,\mu)ds + \int_s^t Y(t,s,\mu)f(s,\mu)ds \,. \quad (1.4.25)$$

Using the relations (1.4.3), (1.1.26) from (1.4.25) can be easily obtained integral equation, which is equivalent to the differential equation (1.1.25)

$$\tilde{y}\left(t,\mu\right) = G\left(t,t_0,\mu\right)\tilde{y}\left(s,\mu\right) + \int_s^t G\left(t,\sigma,\mu\right)\tilde{B}\left(\sigma,\mu\right)u\left(\sigma,\mu\right)d\sigma + \int_s^t G\left(t,\sigma,\mu\right)\tilde{f}\left(\sigma,\mu\right)d\sigma \quad (1.4.26)$$

where $\tilde{B}\left(t,\mu\right) = M^{-1}\left(t,\mu\right)B\left(t,\mu\right), \quad \tilde{f}\left(t,\mu\right) = M^{-1}\left(t,\mu\right)f\left(t,\mu\right), \quad \tilde{y}^0 = \tilde{y}(t_0)$, is transition matrix $G(t,t_0,\mu)$ determined from (1.4.4).

Now suppose that the matrices $\Phi(t,t_0,\mu)$ and $\Psi(t,t_0,\mu)$ are transitional matrix of the system (1.1.22) and (1.1.23). Along with the system (1.1.22) and (1.1.23), we consider another system

$$\dot{\bar{x}} = A_0(t)\bar{x} + B_0(t)\bar{u} + f_0\left(t\right), \;\; \bar{x}(t_0) = \bar{x}^0, \;\; \bar{x}(t_1) = \bar{x}^1, \quad (1.4.27)$$

$$\mu \dot{\bar{z}}_* = A_4(t)\bar{z}_* + B_2(t)\bar{u} + f_2(t),\ \ \bar{z}_*(t_0) = \bar{z}_*^0,\ \bar{z}_*(t_1) = \bar{z}_*^1,$$

where

$$A_0(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t),\quad B_0(t) = B_1(t) - A_2(t)A_4^{-1}(t)B_2(t),$$

$$f_0(t) = f_1(t) - A_2(t)A_4^{-1}(t)f_2(t),\ \ \dot{z}_* = \bar{z} + A_4^{-1}A_3\bar{x}, \tag{1.4.28}$$

$\bar{x}, \bar{z}$ - vectors of state variables of the degenerate system, which is obtained from (1.1.22) and (1.1.23) at $\mu = 0$. Have the following theorem.

Theorem 1.4.2. Let $A_0(t),\ \ A_4(t)$ - stable matrices and corresponding to them transition matrices satisfy the inequalities

$$\left\|\bar{\Phi}(t,t_0)\right\| \le c\exp\left(-m(t-t_0)\right),\ \left\|\bar{\Psi}(t,t_0,\mu)\right\| \le c\exp\left(-\gamma(t-t_0)/\mu\right). \tag{1.4.29}$$

Then at m>1, $0 < \mu < \mu_0 < 1$ and $t_0 \le t \le t_1$ the eigenvalues value matrices $\tilde{A}_1(t,\mu), \tilde{A}_4(t,\mu)$ will be «close» to the eigenvalues of the matrices $A_0(t),\ A_4(t)$, in the sense of negativity their real parts, where

$$\mu_0 = \min\left\{\frac{1}{d_2 c},\quad \frac{\gamma}{d_1 c}\right\}, \tag{1.4.30}$$

$$d_1 = \max_{t_0 \le s \le \sigma \le t \le t_1} L_1\left\|A_2(\sigma)\right\|,\ d_2 = \max_{t_0 \le s \le \sigma \le t \le t_1} L_2\left\|A_2(\sigma)\right\|, \tag{1.4.31}$$

At the same time we have the estimates:

$$\left\|\Phi(t,t_0,\mu)\right\| \le c\exp\left(-m(t-t_0)\right),\ \left\|\Psi(t,t_0,\mu)\right\| \le c\exp\left(-\gamma(t-t_0)/\mu\right), \tag{1.4.32}$$

where the $m_1 = m-1,\ \gamma_1 = \gamma - \mu d_1 c,\ \ c, m, \gamma$ - positive constants,

$\bar{\Phi}(t,t_0), \bar{\Psi}(t,t_0,\mu)$ - transition matrices slow and fast subsystems (1.4.27).

Proof. Let

$$H(t,\mu) = -A_4^{-1}(t)A_3(t) + \mu h(t,\mu)\,, \tag{1.4.33}$$

where the $h(t,\mu)$ satisfies the matrix equation

$$\mu\dot{h}(t,\mu) + \mu h(t,\mu)A_0^*(t,\mu) = A_3^*(t,\mu) + A_4^*(t,\mu)h(t,\mu),$$
$$A_0^*(t,\mu) = A_0(t) + \mu A_2(t)h(t,\mu),$$

$$A_3^*(t) = \frac{d}{dt}(A_4^{-1}(t)A_3(t)) + A_4^{-1}(t)A_3(t)A_0(t), \;\; A_4^*(t,\mu) = A_4(t) + \mu A_4^{-1}(t)A_3(t)A_2(t)\,.$$

In view of (1.4.33), matrices $\tilde{A}_1(t,\mu), \tilde{A}_4(t,\mu)$ are defined as

$$\tilde{A}_1(t,\mu) = A_0(t) + \mu A_2(t)h(t,\mu), \tag{1.4.34}$$

$$\tilde{A}_4(t,\mu) = A_4(t) + \mu A_4^{-1}(t)A_3(t)A_2(t) - \mu^2 h(t,\mu)A_2(t,\mu).$$

As shown in 1.2, matrix $H(t,\mu)$ is the solution of the integral equation (1.2.1) and defined as the limit of a sequence of continuous functions in a closed interval $[t_0,t_1]$, then can specify the number of μ_1 such that $0<\mu\le\mu_1$ in the interim $[t_0,t_1]$ there are limitations:

$$\|H(t,\mu)\| \le L_1, \;\; \|h(t,\mu)\| \le L_2, \tag{1.4.35}$$

where the L_1, L_2 - positive numbers. Transition matrices $\Phi(t,t_0,\mu)$ and $\Psi(t,t_0,\mu)$ may be represented in the form:

$$\Phi(t,t_0,\mu) = \bar{\Phi}(t,t_0) + \mu\varphi(t,t_0,\mu), \;\; \Psi(t,t_0,\mu) = \bar{\Psi}(t,t_0,\mu) + \mu\eta(t,t_0,\mu), \tag{1.4.36}$$

where the $\varphi(t,t_0,\mu)$, $\eta(t,t_0,\mu)$ - matrices functions, $\bar{\Phi}(t,t_0), \bar{\Psi}(t,t_0,\mu)$ - transition matrices slow and fast subsystems (1.4.27).

The process of determining the functions $\varphi(t,t_0,\mu)$ and $\eta(t,t_0,\mu)$ leads us to clarify the main question: for sufficiently small values of the parameter μ, the eigenvalues of matrices $\tilde{A}_1(t,\mu)$ and $\tilde{A}_4(t,\mu)$ will indeed be close to the eigenvalues of the matrix $A_0(t)$ and $A_4(t)$ respectively.

We assume that $A_0(t)$ and $A_4(t)$ - stable matrix. Then the corresponding transition matrices satisfy the inequalities (1.4.29). By assumption, the matrix $\bar{\Phi}(t,t_0)$ and $\bar{\Psi}(t,t_0,\mu)$ satisfy the equations:

$$\dot{\bar{\Phi}}(t,t_0) = A_0(t)\Phi(t,t_0),\ \dot{\Phi}(t_0,t_0) = E_n, \tag{1.4.37}$$

$$\mu\dot{\bar{\Psi}}(t,t_0,\mu) = A_4(t)\bar{\Psi}(t,t_0,\mu),\ \bar{\Psi}(t_0,t_0,\mu) = E_m / \mu. \tag{1.4.38}$$

Then, taking into account (1.4.33), (1.4.36) from the (1.4.1) and (1.4.2), obtain

$$\dot{\varphi}(t,t_0,\mu) = A_0(t)\varphi(t,t_0,\mu) + A_2(t)h(t,\mu)\left(\bar{\Phi}(t,t_0) + \mu\varphi(t,t_0,\mu)\right),$$
$$\varphi(t_0,t_0,\mu) = 0, \tag{1.4.39}$$

$$\mu\dot{\eta}(t,t_0,\mu) = A_4(t)\eta(t,t_0,\mu) - H(t,\mu)A_2(t)\left(\bar{\Psi}(t,t_0,\mu) + \mu\eta(t,t_0,\mu)\right),$$
$$\eta(t_0,t_0,\mu) = 0. \tag{1.4.40}$$

Equations (4.1.39) and (4.1.40) are equivalent to the following integral equation:

$$\varphi(t,t_0,\mu) = \int_{t_0}^{t} \bar{\Phi}(t,\sigma)A_2(\sigma)h(\sigma,\mu)\left(\bar{\Phi}(\sigma,t_0) + \mu\varphi(\sigma,t_0,\mu)\right)d\sigma, \tag{1.4.41}$$

$$\eta(t,t_0,\mu) = \frac{1}{\mu}\int_{t_0}^{t} \bar{\Psi}(t,\sigma,\mu)H(\sigma,\mu)A_2(\sigma)\left(\bar{\Psi}(\sigma,t_0,\mu) + \mu\eta(\sigma,t_0,\mu)\right)d\sigma. \tag{1.4.42}$$

We now define two sequences:

$$\varphi_0(t,t_0,\mu) = \int_{t_0}^{t} \bar{\Phi}(t,\sigma)A_2(\sigma)h(\sigma,\mu)\bar{\Phi}(\sigma,t_0)d\sigma, \tag{1.4.43}$$

$$\varphi_k(t,t_0,\mu)=\varphi_0(t,t_0,\mu)+\mu\int_{t_0}^{t}\bar{\Phi}(t,\sigma)A_2(\sigma)h(\sigma,\mu)\varphi_{k-1}(\sigma,t_0,\mu)d\sigma,$$

$$\eta_0(t,t_0,\mu)=\frac{1}{\mu}\int_{t_0}^{t}\bar{\Psi}(t,\sigma,\mu)H(\sigma,\mu)A_2(\sigma)\bar{\Psi}(\sigma,t_0,\mu)d\sigma \qquad 1.4.44)$$

$$\eta_k(t,t_0,\mu)=\eta_0(t,t_0,\mu)+\int_{t_0}^{t}\bar{\Psi}(t,\sigma,\mu)H(\sigma,\mu)A_2(\sigma)\eta_{k-1}(\sigma,t_0,\mu)d\sigma.$$

Investigate the convergence of the sequences (1.4.43) and (4.1.44).

We introduce the notation in the form (1.4.31). Now we estimate the total members of the following series:

$$\varphi_0+\sum_{n=1}^{\infty}\left(\varphi_n-\varphi_{n-1}\right),\;\; \eta_0+\sum_{k=1}^{\infty}\left(\eta_n-\eta_{n-1}\right). \qquad (1.4.45)$$

In view of (1.4.31) from the (1.4.43) obtain

$$\|\varphi_0\|\le d_2c^2(t-t_0)\exp\left(-m(t-t_0)\right),$$
$$\|\varphi_1-\varphi_0\|\le \mu d_2^2c^3\left((t-t_0)^2/2!\right)\exp\left(-m(t-t_0)\right).$$

By induction, we obtain the inequality

$$\|\varphi_n-\varphi_{n+1}\|\le \mu^n d_2^{n+1}c^{n+2}\left((t-t_0)^{n+1}/(n+1)!\right)\exp\left(-(t-t_0)\right).$$

Of these estimates imply that the first row of (1.4.45) converges at $\mu<1/(d_2c)$, and evenly and as a majorant series supports a number of

$$d_2c^2\left(t-t_0+(t-t_0)/2!+(t-t_0)^3/3!+\ldots+(t-t_0)^n/n!+\ldots\right)\exp\left(-m(t-t_0)\right),$$

which is the sum of $d_2c^2\left(\exp\left(t-t_0\right)-1\right)\exp\left(-m(t-t_0)\right)$.

Consequently, there $\varphi(t,t_0)$ - limit of the first series (1.4.45) exists. Function $\phi(t,t_0)$ satisfies to the equation (1.4.41) and it satisfies the inequality

$$\|\varphi\| \leq d_2 c^2 \left(\exp\left(t - t_0\right) - 1\right) \exp\left(-m(t - t_0)\right), \qquad (1.4.46)$$

At $\mu < 1/(d_2 c)$.

Similarly, from (1.4.37), we have $\|\eta_0\| \leq (1/\mu) d_1 c^2 \exp\left(-(\gamma(t - t_0))/\mu\right)$ and $\|\eta_n - \eta_{n-1}\| \leq (1/\mu) d_1^{n+1} c^{n+2} \left((t - t_0)^{n+1}/(n+1)!\right) \exp\left(-(\gamma(t - t_0)/\mu\right)$.

Then there $\eta(t, t_0, \mu)$ - limit of the second row (1.4.45) exists. The limit function $\eta(t, t_0, \mu)$ at $\mu > 0$ is a solution equation (1.4.44) and in this case we obtain the estimate

$$\|\eta\| \leq (c/\mu)\left(\exp\left(d_1 c(t - t_0)\right) - 1\right) \exp\left(-\left(\gamma(t - t_0)\right)/\mu\right). \qquad (1.4.47)$$

Now, using (14.29) and (1.4.46), (1.4.47) from the (1.4.36), obtain (1.4.32).

Chapter 2

Research Controllability and Dynamics of Movement Singularly Perturbed System

In this chapter is formulated the criterion of controllability using properties of the operator Gramm, and to deal with the evaluation of the standard deviation of the trajectory of motion of the system.

2.1 Controllability Singularly Perturbed Systems of Optimal Control with Constantly Acting External Forces

Here is investigated the properties of controllability of the system (2.1.1) with the help of operator Gram transforming infinite space in finite. Let the controlled process is described by the equation

$$\dot{y} = A(t,\mu)y + B(t,\mu)u + f(t,\mu), \tag{2.1.1}$$

$$y(t_0,\mu) = y^0 \tag{2.1.2}$$

$$y(t_1,\mu) = y^1 \tag{2.1.3}$$

where $A(t,\mu) = \begin{pmatrix} A_1(t) & A_2(t) \\ \dfrac{A_2(t)}{\mu} & \dfrac{A_3(t)}{\mu} \end{pmatrix}$, $B(t,\mu) = \begin{pmatrix} B_1(t) \\ \dfrac{B_2(t)}{\mu} \\ \end{pmatrix}$, $f(t,\mu) = \begin{pmatrix} f_1(t) \\ \dfrac{f_2(t)}{\mu} \\ \end{pmatrix}$,

$y = \begin{pmatrix} x \\ z \end{pmatrix} \in R^{n+m}$ $x \in R^n$, $z \in R^m$ - state vectors, $n \in C^k[t_0,t_1]$, ($C^k[t_0,t_1]-$ infinite space), $f_1(t) \in R^n$, $f_2(t) \in R^m$ constantly operating outside forces; $t \in [t_0,t_1]$, $\mu > 0 -$ small parameter $(0 < \mu \le 1)$.

States $x = x(t,\mu)$, $z = z(t,\mu)$ are slow and fast motion of the system (2.1.1), respectively. We assume the following assumptions regarding the parameters of the system (2.1.1):

1. Matrix $A_i(t)$ $\left(i=\overline{1,4}\right)$ - identified uniformly bounded and uniformly continuous with their derivatives.
2. All eigenvalues of the matrix $A_4(t)$ have negative real parts for all $t\in[t_0,t_1]$.

For linear systems usually criterion controllability are formulated using the properties of a linear operator [4].

First, consider the simplest case when the matrix $A(t,\mu)$ in the equation of the system (2.1.1) is equal to zero matrix. Then the dynamics of the system described by the equation

$$y=B(t,\mu)u+f(t,\mu). \tag{2.1.4}$$

We pose the problem of the choice of control $u(t)=u(t,\mu)$, which would ensure at the time satisfy the boundary conditions (2.1.3). Considering the conditions (2.1.2), (2.1.3) from the equation of motion (2.1.4) obtain the

$$y^1=y^0+\int_{t_0}^{t_1}B(t,\mu)u(s,\mu)ds+\int_{t_0}^{t_1}f(s,\mu)ds. \tag{2.1.5}$$

Then the expression $L(u)=\int_{t_0}^{t_1}B(t,\mu)u(t,\mu)dt$ can be viewed as a linear operator acting from the space $C^m[t_0t_1]$ to R^{n+m}. Because is required choose the control $u(t,\mu)$, which would satisfy the condition (2.1.5), it is easy to see, that if $y^1-y^0-\int_{t_0}^{t_1}f(s,\mu)ds$ lies in the region of the operator $L(u)$, then the desired transition to the state $y(t_1)=y^1$ available. Otherwise - is not. Therefore, to check whether state-controlled necessary to establish, whether it is in the region values of the operator $L(u)$.

Control $u(t)=u(t,\mu)$, which transfers status of the system (2.1.4) from y^0 at $t=t_0$ to y^1 exists only when the vector $y^1-y^0-\int_{t_0}^{t_1} f(s,\mu)ds$ lies in the region values of a linear transformation

$$W(t_0,t_1,\mu)=\int_{t_0}^{t_1} B(s,\mu)B'(s,\mu)ds \qquad (2.1.6)$$

At one of the controls that translates system from one state into another and has the form:

$$u(t,\mu)=B'(t,\mu)\eta \qquad (2.1.7)$$

where η is any solution of the equation

$$W(t_0,t_1,\mu)\eta = y^1-y^0-\int_{t_0}^{t_1} f(s,\mu)ds\,. \qquad (2.1.8)$$

Now we move to a system of general form (2.1.1) when $A(t,\mu)\neq 0$. Integrating the equations of motion of the system (2.1.1) gives

$$y(t,\mu)=Y(t,t_0,\mu)y^0+\int_{t_0}^{t_1} Y(t,s,\mu)B(s,\mu)u(s,\mu)ds+\int_{t_0}^{t_1} Y(t,s,\mu)f(s,\mu)ds, \qquad (2.1.9)$$

where $Y(t,s,\mu)$ - transition matrix for the equation

$$\dot{y}=A(t,\mu)y, \quad y(t_0,\mu)=y^0 \qquad (2.1.10)$$

At $t=t_1$, taking into account (2.1.3) from (2.1.9) we will have equation of moment [42].

$$\alpha(\mu)=\int_{t_0}^{t_1} Y(t_0,s,\mu)B(s,\mu)u(s,\mu)ds \tag{2.1.11}$$

where $\alpha(\mu)=y^0-Y(t_0,t_1,\mu)y'-\int_{t_0}^{t_1} Y(t_0,s,\mu)f(s,\mu)ds.$

Theorem 2.1.1. For system (2.1.1) if and only if exists a control $u(t)=u(t,\mu)$, which transfers from state of the system (2.1.2) to the state (2.1.3) at $t=t_1>t_0$, when the vector $\alpha(\mu)$ belongs in the field of the values of the linear transformation

$$W(t_0,t_1,\mu)=\int_{t_0}^{t_1} Y(t_0,s,\mu)B(s,\mu)B'(s,\mu)Y(t_0,s,\mu)ds\,. \tag{2.1.12}$$

At that control

$$u(t,\mu)=-B'(s,\mu)Y(t_0,t_1,\mu)y_* \tag{2.1.13}$$

is one of the controls to ensure this transition, where the vector y_* is determined from the equation

$$W(t_0,t_1,\mu)y_*=\alpha(\mu). \tag{2.1.14}$$

Proof. We introduce the change of variable

$$\eta(t,\mu)=Y(t_0,t,\mu)y(t,\mu). \tag{2.1.15}$$

Then by the properties of the transition matrix will be

$$y(t,\mu)=Y(t,t_0,\mu)\eta(t,\mu),$$

$$\begin{aligned}&Y(t,t_0,\mu)\dot{\eta}(t,\mu)+\dot{Y}(t,t_0,\mu)\eta(t,\mu)\\&=A(t,\mu)Y(t,t_0,\mu)\eta(t,\mu)+B(t,\mu)u(t,\mu)+f(t,\mu)\end{aligned}$$

or

$$Y(t,t_0,\mu)\dot{\eta}(t,\mu)=B(t,\mu)u(t,\mu)+f(t,\mu).$$

Multiplying this equality on the left to matrix $Y(t_0,t,\mu)$ obtain the

$$\dot{\eta}(t,\mu)=Y(t_0t,\mu)B(t,\mu)u(t,\mu)+Y(t_0,t,\mu)f(t,\mu). \tag{2.1.16}$$

If reasoning as in the previous case, control $u(t,\mu)$ exists if and only if the set of values that can take the

$$\eta(t_1,\mu)-\eta(t_0,\mu)-\int_{t_0}^{t_1}Y(t_0,s,\mu)f(s,\mu)ds$$

belongs to the region of values of operator

$$W(t_0,t_1,\mu)=\int_{t_0}^{t_1}Y(t_0,s,\mu)B(s,\mu)B'(s,\mu)Y'(t_0,s,\mu)ds. \tag{2.1.17}$$

Then the desired transition is possible, if we to require that there has been a

$$\eta(t_1,\mu)-\eta(t_0,\mu)-\int_{t_0}^{t_1}Y(t_0,s,\mu)f(s,\mu)ds$$
$$=Y(t_0,t_1,\mu)y^1-y^0-\int_{t_0}^{t_1}Y(t_0,s,\mu)f(t,\mu)ds=-\alpha(\mu).$$

This means that the desired transformation is possible if and only if the vector $\alpha(\mu)$ for each $\mu>0$ lies in the region values $W(t_0,t_1,\mu)$ and one of the control providing this transformation is a control (2.1.13), q.e.d.

It follows from this theorem that if $0<\mu<1$ and a t_0 for all t_1 matrix $W(t_0,t_1,\mu)$ has maximal rank, then the system (2.1.1) is completely controllable. Matrix $W(t_0,t_1,\mu)$ in shape (2.1.12) at $\mu>0$ has the following properties [65]: it is symmetric, non-negative, defined for $t_1\geq t_0$ and satisfies:

a) matrix differential equation

$$\begin{aligned}&\dot{W}(t,t_1,\mu)=A(t,\mu)W(t,t,\mu)+W(t,t_1,\mu)A'(t,\mu)-B(t,\mu)B'(t,\mu),\\&W(t_1,t_1,\mu)=0\end{aligned}\qquad(2.1.18)$$

b) functional equation

$$W(t_0,t_1,\mu)=W(t_0,t,\mu)+Y(t_0,t,\mu)W(t,t_1,\mu)Y'(t_0,t,\mu).\qquad(2.1.19)$$

If we introduce in the form of a block matrix

$$W(t,t_1,\mu)=\begin{pmatrix}W_1(t,t_1,\mu) & W_2(t,t_1,\mu)\\ W_2^{'}(t,t_1,\mu) & \dfrac{1}{\mu}W_3(t,t_1,\mu)\end{pmatrix},\qquad(2.1.20)$$

then the equation (2.1.18) can be rewritten as a system of three linear singularly perturbed equations are not separated variables:

$$\begin{aligned}&\dot{W}_1=A_1(t)W_1+A_2(t)W_2^{'}+W_1^{'}A_1^{'}(t)+W_2A_2(t)-B(t)B_1^{'}(t),\\&\mu\dot{W}_2=\mu A_1(t)W_2+A_2(t)W_3+W_1A_3^{'}(t)+W_2A_4^{'}(t)-B_1(t)B_2^{'}(t),\\&\mu\dot{W}_3=\mu A_3(t)W_2+A_4(t)W_3+\mu W_2^{'}A_3^{'}(t)+W_3A_4^{'}(t)+B_2(t)B_2^{'}(t),\end{aligned}\qquad(2.1.21)$$

$$W_1(t_1,t_1,\mu)=0,\ W_2(t_1,t_1,\mu)=0,\ W_1(t_1,t_1,\mu)=0\qquad(2.1.22)$$

Theorem 2.1.2. Let matrix $H=H(t,\mu)$, $N=N(t,\mu)$ are solutions of differential equations

$$-\mu\dot{H}-\mu H\big(A_1(t)+A_2(t)H\big)+A_3(t)+A_4(t)H=0,\qquad(2.1.23)$$

$$\mu\dot{N}-\mu H\big(A_1(t)+A_2(t)H\big)N+N\big(A_4(t)-\mu HA_2(t)\big)+A_2(t)=0.\qquad(2.1.24)$$

Then the matrix

$$\tilde{W}(t,t_1,\mu)=\int_t^{t_1} G(t,s,\mu)\tilde{B}(s,\mu)\tilde{B}'(s,\mu)G'(t,s,\mu)ds \tag{2.1.25}$$

satisfies the matrix differential equation

$$\dot{\tilde{W}}=\tilde{A}(t,\mu)\tilde{W}+\tilde{W}\tilde{A}_1'(t,\mu)-\tilde{B}(t,\mu)\tilde{B}_1'(t,\mu), \tag{2.1.26}$$

where

$$\tilde{A}(t,\mu)=M^{-1}(t,\mu)\big(A(t,\mu)M(t,\mu)-\dot{M}(t,\mu)\big), \tag{2.1.27}$$

$$\tilde{B}(t,\mu)=M^{-1}(t,\mu)B(t,\mu),$$

$$M(t,\mu)=\begin{pmatrix} E_n & -\mu N(t,\mu) \\ H(t,\mu) & E_m-\mu H(t,\mu)N(t,\mu)\end{pmatrix},$$

$$G(t,s,\mu)=\begin{pmatrix} \Phi(t,s,\mu) & 0 \\ 0 & \Psi(t,s,\mu)\end{pmatrix},$$

$\Phi(t,s,\mu),\Psi(t,s,\mu)-$ transition matrices of homogeneous equations:

$\dot{\tilde{x}}=\big(A_1(t)+A_2(t)H(t,\mu)\big)x, \quad \mu\dot{\tilde{z}}=\big(A_4(t)-\mu H(t,\mu)A_2(t)\big)\tilde{z}$ respectively.

Proof. In the matrix equation (2.1.18) we introduce the change of variables in the form of

$$W=M(t,\mu)\tilde{W}M'(t,\mu). \tag{2.1.28}$$

Then in view of (2.1.28) from (2.1.18) we have

$$\dot{M}\tilde{W}M'+M\dot{\tilde{W}}M'+M\tilde{W}M'=AM\tilde{W}M'+M\tilde{W}M'A'+BB',$$

$$M\dot{\tilde{W}}M'=\big(AM-\dot{M}\big)\tilde{W}M'+M\tilde{W}\big(M'A'-\dot{M}'\big)-BB'.$$

Multiplying the left by the matrix M^{-1} and the right to M'^{-1} we obtain the equation (2.1.26). When the condition of the theorem matrix

$$\tilde{A}(t,\mu) = M^{-1}(t,\mu)\left(A(t,\mu)M(t,\mu) - \dot{M}(t,\mu)\right)$$

is a diagonal block matrix, i.e.

$$\tilde{A}(t,\mu) = \begin{pmatrix} A_1(t) - A_2(t)H(t,\mu) & 0 \\ 0 & \dfrac{1}{\mu}(A_4(t) - \mu H(t,\mu)A_2(t)) \end{pmatrix}.$$

We calculate the derivative of the function $\tilde{W}(t,t_1,\mu)$ by t

$$\dot{\tilde{W}} = \frac{d}{dt}\left(\int_t^{t_1} G(t,s,\mu)\tilde{B}(s,\mu)\tilde{B}'(s,\mu)G'(t,s,\mu)ds\right) =$$

$$-\tilde{B}(t,\mu)\tilde{B}'(t,\mu) + \tilde{A}(t,\mu)\int_t^{t_1} G(t,s,\mu)\tilde{B}(s,\mu)B'(s,\mu)G'(t,s,\mu)ds +$$

$$+\int_t^{t_1} G(t,s,\mu)B(s,\mu)\tilde{B}'(s,\mu)G'(t,s,\mu)ds\tilde{A}'(t,\mu)$$

$$= \tilde{A}(t,\mu)\tilde{W} + \tilde{W}A'(t,\mu) - \tilde{B}(t,\mu)B'(t,\mu)$$

q.e.d.

As in the previous case, if you enter the block matrix

$$\tilde{W}(t,t_1,\mu) = \begin{pmatrix} \tilde{W}_1(t,t_1,\mu) & \tilde{W}_2(t,t_1,\mu) \\ \tilde{W}_2'(t,t_1,\mu) & \dfrac{1}{\mu}\tilde{W}_3(t,t_1,\mu \end{pmatrix},$$

then the equation (2.1.26) can be rewritten as a system of singularly perturbed three equations with separated variables

$$\dot{\tilde{W}}_1 = \tilde{A}_1(t,\mu)\tilde{W}_1 + \tilde{W}_1\tilde{A}_1'(t,\mu) - B_1(t,\mu)\tilde{B}_1'(t,\mu),$$

$$\mu\dot{\tilde{W}}_2 = \mu\tilde{A}_1(t,\mu)\tilde{W}_2 + \tilde{W}_2'\tilde{A}_4'(t,\mu) - \tilde{B}_1(t,\mu)\tilde{B}_2'(t,\mu), \qquad (2.1.29)$$

$$\mu\dot{\tilde{W}}_3 = \tilde{A}_4(t,\mu)\tilde{W}_3 + \tilde{W}_3\tilde{A}_4'(t,\mu) - \dot{\tilde{B}}_2(t,\mu)\tilde{B}_2'(t,\mu),$$

with the final conditions

$$\tilde{W}_1(t_1,t_1,\mu)=0,\ \tilde{W}_2(t_1,t_1,\mu)=0,\ \tilde{W}_3(t_1,t_1,\mu)=0, \qquad (2.1.30)$$

where $\tilde{A}_1(t,\mu) = A_1(t) + A_2(t)H(t,\mu),\ \ \tilde{A}_4(t,\mu) = A_4(t) + \mu H(t,\mu)A_2(t),$

$$\tilde{B}_1(t,\mu) = B_1(t) + N(t,\mu)\big(B_2(t) - \mu H(t,\mu)B_1(t)\big),$$
$$\tilde{B}_2(t,\mu) = B_2(t) + \mu H(t,\mu)B_1(t).$$

Equations included in the system (2.1.29) does not depend on each other and their solutions are matrix

$$\tilde{W}_1(t,t_1,\mu) = \int_t^{t_1} \Phi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_1(s,\mu)\Phi'(t,s,\mu)ds,$$

$$\tilde{W}_2(t,t_1,\mu) = \frac{1}{\mu}\int_t^{t_1} \Phi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_2'(s,\mu)\Psi'(t,s,\mu)ds,$$

$$\tilde{W}_3(t,t_1,\mu) = \frac{1}{\mu}\int_t^{t_1} \Psi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_2'(s,\mu)\Psi(t,s,\mu)ds,$$

at $\mu \to 0$ for matrix $\tilde{W}_1(t,t,\mu),\ \tilde{W}_2(t,t,\mu),\ \tilde{W}_3(t,t,\mu)$ we have the following limit relations:

$$\tilde{W}_1(t,t_1,\mu), \to \overline{W}_1(t),\ \ \tilde{W}_2(t,t_1,\mu), \to \overline{W}_2(t),\ \ \tilde{W}_3(t,t_1,\mu), \to \overline{W}_3(t),$$

uniformly in $t \in \left[t_0, t_1^*\right] \subset \left[t_0, t_1\right]$. Matrix $\overline{W}_1(t) = \int_t^{t_1} \overline{\Phi}(t,s) B_0(s) B_0'(s) \overline{\Phi}(t,s)\, ds$ is the solution of the matrix differential equation $\dot{\overline{W}} = A_0(t)\overline{W}_1 + \overline{W}_1 A_0'(t) - B_0(t) B_0(t),\; W(t_1, t_1) = 0,$

where $A_0(t) = A_1(t) - A_2(t) A_4^{-1}(t) A_3(t),\; B_0(t) = B_1(t) - A_2(t) A_4^{-1}(t) B_2(t),$

$$\overline{W}_2(t) = B_0(t) B_2'(t) A_4'^{-1}(t),$$

$$\overline{W}_3(t) = \int_0^\infty e^{-A_4(t)\sigma} B_2(t_1) B_2'(t) e^{-A_4''(t)\sigma} d\sigma,$$

is the solution of algebraic equations

$$A_4(t) W_3(t) - \overline{W}(t)_3\, A_4'(t) = B_2(t) B_2'(t).$$

2.2 The Criterion Controllability of Movement of Singularly Perturbed System

As shown in the preceding paragraph, after transformation gramiana controllability we received matrix (2.1.25). The structure of the matrix is not changed and as gramiana controllability can take the matrix (2.1.25). As in the previous case, if enter the block matrix

$$\tilde{W}(t, t_1, \mu) = \begin{pmatrix} \tilde{W}_1(t, t_1, \mu) & \tilde{W}_2(t, t_1, \mu) \\ \tilde{W}_2'(t, t_1, \mu) & \dfrac{1}{\mu}\tilde{W}_3(t, t_1, \mu) \end{pmatrix}, \tag{2.2.1}$$

then the equation (2.1.26) can be rewritten as a system of singularly perturbed three equations with separated variables

$$\begin{aligned}
\dot{\tilde{W}}_1 &= \tilde{A}_1(t,\mu)\tilde{W}_1 + \tilde{W}_1 A_1'(t,\mu) - \tilde{B}_1(t,\mu)\tilde{B}_1'(t,\mu),\\
\mu\,\dot{\tilde{W}}_2 &= \mu\,\tilde{A}_1(t,\mu)\tilde{W}_2 + \tilde{W}_2'\tilde{A}_4'(t,\mu) - \tilde{B}_1(t,\mu)\tilde{B}_2'(t,\mu)\\
\mu\,\dot{\tilde{W}}_3 &= \tilde{A}_3(t,\mu)\tilde{W}_3 + \tilde{W}_3\tilde{A}_4(t,\mu) - \tilde{B}_2(t,\mu)\tilde{B}_2'(t,\mu),
\end{aligned} \qquad (2.2.2)$$

$$\tilde{W}_1(t_1,t_1,\mu)=0, \quad \tilde{W}_2(t_1,t_1,\mu)=0, \quad \tilde{W}_3(t_1,t_1,\mu)=0, \qquad (2.2.3)$$

where $\tilde{W}_1=\tilde{W}_1(t_1,t_1,\mu)$, $\tilde{W}_3=\tilde{W}_3(t_1,t_1,\mu)-$ symmetric matrices sizes $n\times n$ and $m\times m$ respectively, $\tilde{W}_2=\tilde{W}_2(t_1,t_1,\mu)-$ matrix size $n\times m$,

$$\tilde{A}_1(t,\mu)=A_1(t)+A_2(t)\mathrm{H}(t,\mu), \quad \tilde{A}_4(t,\mu)=A_4(t)+\mu\mathrm{H}(t,\mu)A_2(t),$$

$$\begin{aligned}
\tilde{B}_1(t,\mu) &= B_1(t)+\mathrm{N}(t,\mu)\left(B_2(t)-\mu\mathrm{H}(t,\mu)B_1(t)\right),\\
\tilde{B}_2(t,\mu) &= B_1(t)+\mu\mathrm{H}(t,\mu)B_1(t).
\end{aligned}$$

It should be noted that under the conditions of theorem 2.1.2 the initial system (2.1.1) can be replaced by an equivalent system (1.1.25). Such a change is possible, since the matrix integral manifolds $H=H(t,\mu)$, $N=N(t,\mu)$ as the solutions of equations (1.1.20), (1.1.21) there are exists and unique (see chap. 1).

Then we can formulate the following theorem (analogous to theorem 2.1.1).

Theorem 2.2.1. For the system (1.1.25) at $\mu>0$ if and only if there exists a control $\tilde{u}(t)=\tilde{u}(t,\mu)$, which transfers the system from the initial state $\tilde{y}(t_{0,}\mu)=\tilde{y}^0$ to the final state $\tilde{y}(t_1,\mu)=\tilde{y}^1$ (see 1.1.24) at $t=t_1>t_0$, when the vector $\tilde{\alpha}(\mu)=M^{-1}(t_0,\mu)\alpha(\mu)$ belongs in the region of values of the linear transformation

$$\tilde{W}(t_0,t_1,\mu)=\int_{t_0}^{t_1} G(t_0,s,\mu)\tilde{B}(s,\mu)\tilde{B}'(s,\mu)G'(t_0,s,\mu)ds. \qquad (2.2.4)$$

At the same time the control

$$\tilde{u}(t,\mu) = -\tilde{B}'(t,\mu)G'(t_0,t,\mu)\tilde{y}_* \tag{2.2.5}$$

is one of the controls providing this transition, where the vector is determined from the equation

$$\tilde{W}(t_0,t_1,\mu)\tilde{y}_* = \tilde{\alpha}(\mu), \tag{2.2.6}$$

where

$$\tilde{\alpha}(\mu) = M^{-1}(t_0,\mu)\alpha(\mu) = M^{-1}(t_0,\mu)y^0 - G(t_0,t_1,\mu)M^{-1}(t_1,\mu)y^1$$
$$+\int_{t_0}^{t_1} G(t_0,s,\mu)M^{-1}(s,\mu)f(s,\mu)ds.$$

As shown by the formula (2.2.6) if the matrix $\tilde{W}(t_0,t_1,\mu)$ has maximal rank, then the control system provides translation (1.1.25) from the initial state $(t_0,\tilde{y}^0)$ to the final state $(t_1,\tilde{y}^1)$ and system (1.1.25) (and simultaneously the system (2.1.1)) is considered quite controllable. Therefore, our the nearest goal is to deduce from the system of equations (2.2.2.) the conditions that provide full controllability of the system (2.1.1.).

For sufficiently small values μ, of the equations obtained with respect $\tilde{W}_2$ and $\tilde{W}_3$ are singularly perturbed. At $\mu = 0$ we have no disturbed (degenerate) system

$$\begin{aligned}
&\dot{\bar{W}}_1 = A_0(t)\bar{W}_1 + \bar{W}_1 A_0'(t) - B_0(t)B_0'(t), \quad \bar{W}_1(t_1,t_1) = 0,\\
&0 = \bar{W}_2 A_4'(t) - B_0(t)B_2'(t),\\
&0 = A_4(t_1)\bar{W}_3 + \bar{W}_3 A_4'(t) - B_2(t)B_2'(t),
\end{aligned} \tag{2.2.7}$$

where $A_0(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t), \quad B_0(t) = B_1(t) - A_2(t)A_4^{-1}(t)B_2(t).$

The solution of the degenerate system approximates the solution of the problem (2.2.2), (2.2.3) with precision $O(\mu)$, and for $\tilde{W}_2$ and $\tilde{W}_3$ this is true outside the boundary layer [45], i.e. at away from the point $(t_1,0)$.

Since we are interested in the values of submatrices $\bar{W}_i \quad (i=2,3)$ at the point $t=t_0$, so the value of $\bar{W}_i \quad (i=2,3)$ at the point $t=t_0$, substitute values of submatrices $\bar{W}_i \quad (i=2,3)$ at indicated the point with an accuracy $O(\mu)$.

At $t=t_0$ from (2.2.7) we have a matrix algebraic equations with constant coefficients. From the second equation can be determined immediately $W_2(t_0,t_1)$:

$$W_2(t_0,t_1)=B_0(t_0)B_2'(t_0)A_4'^{-1}(t_0) \tag{2.2.8}$$

The equation for $W_3(t_0,t_1)$ is the equation of Lyapunov:

$$A_4(t_0)\bar{W}_3(t_0,t_1)+\bar{W}_3(t_0,t_1)A_4'(t_0)=B_2(t_0)B_2'(t_0) \tag{2.2.9}$$

Since the proposal for the real parts of the eigenvalues values of matrix $A_4(t)$ negative for all $t\in[t_0,t_1]$, then the solution of the Lyapunov equation can be represented as a convergent integral [45]

$$\bar{W}_3(t_0,t_1)=\int_0^\infty e^{-A_4(t_0)\tau}B_2(t_0)B_2'(t_0)e^{-A_4'(t_0)\tau}d\tau \tag{2.2.10}$$

solutions (2.2.8) and (2.2.10) may be obtained by other ways. Let’s show it.

Decision matrix equations (2.2.2.) can be formally represented as [45]

$$\tilde{W}_1(t,t_1\mu)=\int_t^{t_1}\Phi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_1'(s,\mu)\Phi'(t,s,\mu)ds, \qquad (2.2.11)$$

$$\tilde{W}_2(t,t_1,\mu)=\frac{1}{\mu}\int_t^{t_1}\Phi(t,s,\mu)\tilde{B}_1(s,\mu)\tilde{B}_2'(s,\mu)\Psi'(t,s,\mu)ds, \qquad (2.2.12)$$

$$\tilde{W}_3(t,t_1,\mu)=\frac{1}{\mu}\int_t^{t_1}\Psi(t,s,\mu)\tilde{B}_2(s,\mu)\tilde{B}_2'(s,\mu)\Psi'(t,s,\mu)ds. \qquad (2.2.13)$$

At $\mu\to 0$ matrix $\tilde{W}_1(t_1,t_0\mu)$ – (2.2.11) tends to the solution of the first equation of the system (2.2.7), i.e.

$$\overline{W_1}(t,t_1)=\int_t^{t_1}\overline{\Phi}(t,s)B_0(s)B_0'(s)\overline{\Phi}'(t,s)ds, \qquad (2.2.14)$$

where $\bar{\Phi}(t,s)$– transition matrix for the homogeneous equation $\bar{x}(t)=A_0(t)\bar{x}(t)$,

$$A_0(t)=A_1(t)-A_2(t)A_4^{-1}(t)A_3(t), \quad B_0(t)=B_1(t)-A_2(t)A_4^{-1}(t)B_2(t).$$

We introduce a new variable $\tau=\dfrac{t-t_0}{\mu}$ to (2.2.12), (2.2.13) we note that at sufficiently small μ matrices $A_4(t_0+\tau\mu)$, $B_0(t_0+\tau\mu)$, $B_2(t_0+\tau\mu)$ are slowly varying functions in the space and they can be replaced by constant matrices $A_4(t_0), B_0(t_0), B_2(t_0)$ [45]. Then at $\mu\to 0$, $(\tau\to\infty)$ for matrix $W_i(t,t_1,\mu)$ $(i=1,2,3)$ at the point $t=t_0$ has the following limit relations:

$$\tilde{W}_1(t_0,t_1,\mu) \to \bar{W}_1(t_0,t_1) = \int_{t_0}^{t_1} \bar{\Phi}(t_0,s) B_0(s) B_0'(s) \bar{\Phi}'(t_0,s)\, ds,$$

$$\tilde{W}_2(t_0,t_1,\mu) \to \bar{W}_2(t_0,t_1) = B_0(t_0) B_2'(t_0) A_4'^{-1}(t_0), \qquad (2.2.15)$$

$$\tilde{W}_3(t_0,t_1,\mu) \to \bar{W}_3(t_0,t_1) = \int_0^{\infty} e^{-A_4(t_0)\tau} B_2(t_0) B_2'(t_0) e^{-A_4'(t_0)\tau}\, d\tau.$$

Lemma. Let matrix $\bar{W}_1$ and $\bar{W}_3$ are nonzero, then at $\mu \to 0$ vector $\tilde{\alpha} = \tilde{\alpha}(\mu)$ will be finite value if and only if the last m components of vector $\tilde{y}_*$ at $\mu \to 0$ tends to zero.

Proof. Let the vector $\tilde{\alpha}$ limited, i.e. exist a number M, that

$$|\tilde{\alpha}_i| \le M \qquad (2.2.16)$$

for all i=1,2…, $n+m$. From (2.2.6) we have the following relation

$$\tilde{W}^{-1}(t_0,t_1,\mu)\tilde{\alpha} = \tilde{y}_* \qquad (2.2.17)$$

Using the formula the Frobenius [13] ratio (2.2.17) is written in the form

$$\begin{pmatrix} \omega_1 & \mu\omega_2 \\ \mu\omega_2' & \mu\omega_3 \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \tilde{x}_* \\ \tilde{z}_* \end{pmatrix}, \qquad (2.2.18)$$

where $\omega_1 = P'$, $\omega_2 = -P^{-1}\tilde{W}_2 \cdot \tilde{W}_3^{-1}$, $\omega_3 = \tilde{W}_3^{-1} - \mu \tilde{W}_3^{-1}\tilde{W}\ P^{-1}\tilde{W}_2\tilde{W}_3^{-1}$,

$P = \tilde{W}_1 - \mu \tilde{W}_2 \tilde{W}_3^{-1} \tilde{W}_2^1$, $\omega_i = \omega_i(t_0,t_1,\mu), i = 1,2,3;$ $\tilde{\alpha}_1, \tilde{x}_* - n -$ dimensional, $\tilde{\alpha}_2, \tilde{z}_* - m -$ dimensional vectors. From (2.2.18) we obtain the

$$\tilde{y}_*(\mu) = \begin{pmatrix} \tilde{x}_* \\ \tilde{z}_* \end{pmatrix} = \begin{pmatrix} \omega_1\tilde{\alpha}_1 + \mu\omega_2\tilde{\alpha}_2 \\ \mu(\omega'_2\,\tilde{\alpha}_1 + \omega_3\tilde{\alpha}_2) \end{pmatrix} \qquad (2.2.19)$$

By the condition of the Lemma, the matrices $\bar{W}_1, \bar{W}_3 -$ are nonzero and reversible. Then for sufficiently small $\mu > 0$ matrix $\omega_i\,(i=1,2,3)$ exist at $\mu \to 0$ from (2.2.19) we have

$$\tilde{y}_*(\mu) \to \bar{y}_*(0) = \begin{pmatrix} \bar{w}_1 \bar{\alpha}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{x}_*(0) \\ 0 \end{pmatrix}.$$

Prove the converse, let the last m components of the vector $\tilde{y}_*(\mu)$ at $\mu \to 0$ tends to zero. It means that m - dimensional vector $\tilde{z}_*(\mu)$ has an estimate

$$\left\|\tilde{z}_*(\mu)\right\| = O(\mu).$$

Then the vector $\tilde{Z}_*(\mu)$ can be represented as

$$\tilde{z}_*(\mu) = \mu\,\beta(\mu),\ \left\|\beta(\mu)\right\| \le M,\ \mathrm{M} - const\ . \tag{2.2.20}$$

Considering (2.2.20) from (2.2.17) we get

$$\tilde{\alpha}(\mu) = \begin{pmatrix} \tilde{W}_1 \tilde{x}_* + \mu\,\tilde{W}_2 \beta \\ \tilde{W}_2' \tilde{x}_* + \tilde{W}_3 \beta \end{pmatrix}.$$

Then for $\mu \to 0$,

$$\tilde{\alpha}(\mu) \to \begin{pmatrix} W_1' \bar{x}_* \\ \bar{W}_2' \bar{x}_* + W_3 \bar{\beta} \end{pmatrix}, \tag{2.2.21}$$

i.e. vector $\tilde{\alpha}(\mu)$ at $\mu \to 0$ is the ultimate value, where $\bar{x}_*, \bar{\beta} - n,\quad m -$ dimensional vectors, respectively, which do not depend on μ. The lemma is proved.

When the condition of the lemma from (2.2.6) we obtain the following equation for the submatrices $\bar{W}_1$ $\bar{W}_3$:

$$\bar{W}_1 \cdot \bar{x}_* = \bar{\alpha}_1 \tag{2.2.22}$$

$$\bar{W}_3 \cdot \bar{\beta} = \bar{\alpha}_2^*, \tag{2.2.23}$$

Where

$$\bar{\alpha}_1 = x^0 - \bar{\Phi}(t_0, t_1)x^1, \quad \bar{\alpha}_2^{\ *} = \bar{\alpha}_2 - \bar{W}_2' \cdot \bar{x}_*,$$
$$\bar{\alpha}_2 = z^0 + A_4^{-1}(t_0)A_3(t_0)x^0 \, ..$$

These relations can be seen at once that for sufficiently small $\mu > 0$, controllability of the two sub-systems of smaller dimension type

$$\dot{\bar{x}}(t) = A_0(t)\bar{x}(t) + B_0(t)u(t) + f_1(t) \quad \mu\dot{\tilde{z}} = A_4(t_0)\tilde{z} + B_2(t_0)u + f_2(t_0), \tag{2.2.24}$$

where $f_0(t) = f_1(t) - \mathrm{A}_2(t)\mathrm{A}_4^{-1}(t) f_2(t)$, should be controllability the complete system (2.1.1). Then from the position of the application of properties of linear operators controllability criterion for the system (2.1.1) is formulated in the following theorem.

Theorem 2.2.2. For the system (2.1.1) if and only if there exists a control $u(t) = u(t, \mu)$, which transfers the system from state (t_0, y^0) to state (t_1, y^1), when vectors $\bar{\alpha}_1 = x^0 - \Phi(t_0, t_1)x^1, \quad \bar{\alpha}_2^{\ *} = \bar{\alpha}_2 - \bar{W}_2{}' \bar{x}_*$ belong to the region of values of linear transformations

$$\bar{W}_1(t_0, t_1) = \int_{t_0}^{t_1} \bar{\Phi}(t_0, s)\mathrm{B}_0(s)\mathrm{B}_0'(s)\bar{\Phi}'(t_0, s)ds, \tag{2.2.25}$$

$$\bar{W}_3\left(t_0,t_1\right)=\int_0^\infty e^{-A_4(t_0)\tau}B_2\left(t_0\right)B_2'\left(t_0\right)e^{-A_4'(t_0)\tau}d\tau. \qquad (2.2.26)$$

Respectively, in addition, if $\bar{x}_*^0$, $\bar{\beta}^0$ - or a solution of (2.2.22) and (2.2.23), it is possible to define control $u=u_0\left(t\right)$, which depending on the time of the partial movements mutually independent sub-systems is described by different analytic expressions and provides this transition to an accuracy $O(\mu)$, i.e. it is written in the form

$$u_0(t)=\begin{cases}\bar{u}^0(t), & t_0\le t\le t_1\\ \bar{u}^0(t_0)+V(\tau), & 0\le\tau\le\dfrac{t_1-t_0}{\mu}<+\infty,\end{cases} \qquad (2.2.27)$$

where $\tau=\dfrac{t-t_0}{\mu}$, $\bar{u}^0\left(t\right)=-B_0'\left(t\right)\Phi'\left(t_0,t\right)x_*^0$, $V\left(\tau\right)=-B_2'\left(t_0\right)e^{-A_4'(t_0)\tau}\beta^0$.

Note that if the vector $\bar{\alpha}_1$ belongs to the region of the linear transformation (2.2.25), the first subsystem of the system (2.2.24) is completely controllable. To prove this part of the theorem is not difficult.

Proof. The prove of the theorem hold for fast subsystem of the system (2.2.24) by means of a change of variable

$$\eta\left(t,\mu\right)=e^{A_4(t_0)\left(\frac{t_0-t}{\mu}\right)}\tilde{z}\left(t,\mu\right). \qquad (2.2.28)$$

Then

$$\tilde{z}\left(t,\mu\right)=e^{A_4(t_0)\left(\frac{t-t_0}{\mu}\right)}\eta\left(t,\mu\right) \qquad (2.2.29)$$

and $\dot{\tilde{z}}\left(t,\mu\right)=e^{A_4(t_0)\left(\frac{t-t_0}{\mu}\right)}\dot{\eta}\left(t,\mu\right)+\dfrac{1}{\mu}A_4\left(t_0\right)e^{A_4(t_0)\left(\frac{t-t_0}{\mu}\right)}\eta\left(t,\mu\right)$.

Substituting the value of $\dot{z}(t,\mu)$ to the fast subsystem (2.2.24) with (2.2.28) and (2.2.29) we obtain

$$\mu e^{A_4(t_0)\left(\frac{t-t_0}{\mu}\right)}\dot{\eta}(t,\mu)=B_2(t_0)u(t)+f_2(t_0),$$

from whence

$$\mu\dot{\eta}(t,\mu)=e^{A_4(t_0)\left(\frac{t_0-t}{\mu}\right)}\left(B_2(t_0)u(t)+f_2(t_0)\right) \qquad (2.2.30)$$

We introduce a new control

$$u(t)=\tilde{u}(t_0)+V(\tau), \qquad (2.2.31)$$

where $\tau=\dfrac{t_0-t}{\mu}$. Then

$$\frac{d\eta}{d\tau}=-e^{-A_4(t_0)\tau}B_2(t_0)\overline{u}^0(t_0)-e^{-A_4(t_0)\tau}B_2(t_0)V(\tau)-e^{-A_4(t_0)\tau}f_2(t_0) \qquad (2.2.32)$$

The solution of the equation can be written as

$$\begin{aligned}\eta(\tau)&=\eta(0)-e^{-A_4(t_0)\tau}A_4^{-1}(t_0)\left(B(t_0)u(t_0)+f_2(t_0)\right)\\&+A_4^{-1}(t_0)\left(B(t_0)u(t_0)+f_2(t_0)\right)-\int_0^\tau e^{-A_4(t_0)S}B_2(t_0)V(s)ds.\end{aligned} \qquad (2.2.33)$$

With the change of variables

$$\begin{aligned}&\eta^*(\tau)=\eta(\tau)+e^{-A_4(t_0)\tau}A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f(t_0)\right),\\&\eta^*(0)=\eta(0)+A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f_2(t_0)\right)\end{aligned}$$

from (2.2.29) we obtain

$$\tilde{z}(\tau)=e^{A_4(t_0)\tau}\eta(\tau)=e^{A_4(t_0)\tau}\eta^*(\tau)-A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f_2(t_0)\right)$$

or $\tilde{z}(\tau)+A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f_2(t_0)\right)=e^{A_4(t_0)}\eta^*(\tau)$, from whence

$$\eta^*(\tau)=e^{-A_4(t_0)\tau}\left[\tilde{z}(\tau)+A_4^{-1}(t_0)\left(B_2(t_0)u(t_0)+f_2(t_0)\right)\right]. \qquad (2.2.34)$$

From the previous lemma is well known that the control $u_0(t)$, which translates the state of the fast subsystem (2.2.24) of the z^0 at $t=t_0$ to z^1 at $t=t_1$ exists if and only if the vector $\eta^*(0)-\eta^*(\tau)$ belongs to the region of values of the matrix $\bar{W}_3(t_0,t_1)$ in (2.2.26).

To complete the desired transition, require that

$$\eta^*(0)-\eta^*(\tau_1)=-\int_0^\infty e^{-A_4(t_0)s}B(t_0)V(s)ds. \qquad (2.2.35)$$

Then one of the controls providing in the unmentioned transition of system has the form

$$V(\tau)=-B_2'(t_0)e^{-A_4'(t_0)\tau}\beta^*, \qquad (2.2.36)$$

where β^* is determined from the equation

$$\eta^*(0)-\eta^*(\tau_1)=\bar{W}_3(t_0,t_1)\beta^*,\ \tau_1=\frac{t_1-t_0}{\mu}.$$

Corollary 1. If the matrices (2.2.25), (2.2.26) have maximal ranks, then the system (2.1.1) is completely controllable.

Corollary 2. In the stationary case:

a) the operator $\bar{W}_1(t_1,t_0)$ (2.2.25) has full rank for any $t_1>t_0$;

b) the operator $\bar{W}_3(t_1,t_0)$ (2.2.26) has full rank if the symmetric matrix B_1B_2' is positive definite.

2.3 Estimation of the Standard Deviation of the Trajectory of the System of Movement

In this section is solved the problem of estimation of the standard deviation of motion of a singularly perturbed system. The main requirement for closed-loop system is the system to return to the zero from any state, and the value of criterion quality along any such motion should be minimized.

Consider the quadratic functional

$$J = \int_{t_0}^{t_1} y'(t)\, w(t)\, y(t)\, dt \tag{2.3.1}$$

where $w(t) = \begin{pmatrix} w_1 & w_2 \\ w_2' & w_3 \end{pmatrix}$.

In the closed-loop optimal trajectory of system is described by homogeneous equations. Therefore avoiding complex analytical expressions and extra notation we restrict homogeneous equations, which are obtained from (2.1.1) at $u(t,\mu) = 0,\ f_1(t,\mu) = 0,\ f_2(t,\mu) = 0.$

By virtue of the equations of motion $\tilde{x}(t) = \Phi(t,t_0\mu)\tilde{x}(t_0), \quad \tilde{z}(t) = \Psi(t,t_0,\mu)\tilde{z}(t_0)$ we have

$$J = \int_{t_0}^{t} \tilde{y}'(t_0) G'(t,t_{0,}\mu) W(t) G(t,t_0\mu)\tilde{y}(t_0)\, dt = \tilde{y}'(t_0) V(t_{0,}t_{1,}\mu)\tilde{y}(t_0), \tag{2.3.2}$$

where

$$G(t,t_{0,}\mu) = \begin{pmatrix} \Phi(t,t_{0,}\mu) & 0 \\ 0 & \Psi(t,t_{0,}\mu), \end{pmatrix},\ V(t,t_0,\mu) = \int_{t_0}^{t} G'(t,t_{0,}\mu)\tilde{W}(t) G(t,t_0\mu)\, dt, \tag{2.3.3}$$

where $\tilde{W} = M'WM, \quad M = \begin{pmatrix} E_n & -\mu N \\ H & E_m - \mu HN \end{pmatrix}.$

Thus, the target value J is a quadratic form $\tilde{y}(t_0)$, and $V(t, t_{0,} \mu)-$ its matrix. If there are known transition matrices $\Phi(t, t_0, \mu)$, $\Psi(t, t_0, \mu)$, then the matrix $V(t_0, t_1, \mu)$ can be calculated using the formula (2.3.3). One can show other methods of calculation. This problem can be reduced to the solution of a linear system with singular perturbations, replacing t_0 to t and differentiating expression for the $V(t, t_1, \mu)$ by t we have:

$$\frac{d}{dt} V(t, t_1, \mu) = \frac{d}{dt}\left(\int_t^{t_1} G'(s, t, \mu) \tilde{W}(s) G(s, t, \mu) ds \right)$$
$$= -\tilde{A}'(t) V(t, t, \mu) - V(t, t_1, \mu) \tilde{A}(t) - \tilde{W}(t). \tag{2.3.4}$$

From the definition $V(t, t_1, \mu)$ it follows that $V(t_1, t_1, \mu) = 0.$

The matrix V is divided into blocks

$$V = \begin{pmatrix} V_1 & \mu V_2 \\ \mu V_2' & \mu V_3 \end{pmatrix} \tag{2.3.5}$$

and the equation (2.3.4) in the form of the system three matrices equations:

$$\dot{V}_1 = -\tilde{A}_1'(t) V_1 - V_1 \tilde{A}_1(t) - \tilde{W}_1(t), \; V_1(t_1, t_1) = 0,$$
$$\mu \dot{V}_2 = -\mu \tilde{A}_1'(t) V_2 - V_2 \tilde{A}_4(t) - \tilde{W}_2(t), \; V_2(t_1, t_1) = 0, \tag{2.3.6}$$
$$\mu \dot{V}_3 = -\tilde{A}_4'(t) V_3 - V_3 \tilde{A}_4(t) - \tilde{W}_3(t), \; V_3(t_1, t_1) = 0,$$

which can be solved independently.

Note that the boundary conditions of differential equations are given at the not initial time but at the end of the process.

Thus, the following theorem holds.

Theorem 2.3.1. If the blocks of matrix V are the solutions of differential equations in (2.3.6), and $\tilde{x}(t)$, $\tilde{z}(t)-$ solutions of the system $\dot{\tilde{x}} = \tilde{A}_1(t)\tilde{x}$, $\mu\dot{\tilde{z}} = \tilde{A}_4(t)\tilde{z}$ at $t_0 \le t \le t_1$, then the formula is true

$$\int_{t_0}^{t_1} \tilde{y}'(t)W(t)\tilde{y}(t)dt = \tilde{y}'(t_0)V(t_0,t_1,\mu)\tilde{y}(t_0), \tag{2.3.7}$$

where $\tilde{y}(t) = \tilde{y}(t,\mu) = \begin{pmatrix} \tilde{x}(t,\mu) \\ \tilde{z}(t,\mu) \end{pmatrix}$.

Limit task (at $\mu \to 0$) for (2.3.6) has the form

$$\bar{V}_1 = -\bar{A}_1(t)\bar{V}_1 - \bar{V}_1\bar{A}_1(t) - \bar{W}_1(t), \quad \bar{V}_1(t_1,t_1) = 0, \tag{2.3.7a}$$

$$0 = -\bar{V}_2 A_4(t) - \bar{W}_2(t) \tag{2.3.7b}$$

$$0 = -A_4'(t)\bar{V}_3 - \bar{V}_3 A_4(t) - \bar{W}_3(t),$$

where $\bar{A}_1(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t)$,

$$\bar{W}_1(t) = W_1 + H_0'W_2' + W_2H_0 + H_0'W_3H_0,$$
$$\bar{W}_2 = W_2 + H_0'W_3, \quad \bar{W}_3 = W_3, \qquad H_0 = -A_4^{-1}A_3.$$

For small μ are possible various ways to construct an approximate solution of the system (2.3.6).

At the basic of the approximate solutions lie solutions "systems of fast movements."

$$\frac{dV_2}{d\tau} = -\mu\bar{A}_1\tilde{V}_2 - \tilde{V}_2A_4 - \bar{W}_2, \qquad \tilde{V}_2(0) = 0, \tag{2.3.8}$$

$$\frac{d\tilde{V}_3}{d\tau} = -A_4'\tilde{V}_3 - \tilde{V}_3A_4 - \bar{W}_3, \qquad \tilde{V}_3(0) = 0, \tag{2.3.9}$$

where $A_0 = A_0\left(t_1\right) \approx \tilde{A}_1\left(t_1 + \tau\mu\right),\ \ \tilde{A}_4\left(t_1 + \tau\mu\right) \approx A_4\left(t_1\right) = A_4,$

$$\bar{W}_i\left(t_1 + \tau\mu\right) \approx \bar{W}_i\left(t_1 + \tau\mu\right) \approx \bar{W}_i\left(t_1\right) = \bar{W}_i \qquad i = \left(2,3\right),\ \ \tau = \frac{t - t_1}{\mu},$$

$$A_0 = A_1 - A_2 A_4^{-1} A_3.$$

Solutions (2.3.8), (2.3.9) are satisfying the zero initial conditions have the form

$$\tilde{V}_2\left(\tau\right) = \int_{\tau}^{0} e^{-\mu\bar{A}_1'(\tau-\delta)_1} \bar{W}_2 e^{-A_4\tau-\sigma} d\sigma, \tag{2.3.10}$$

$$\tilde{V}_3\left(\tau\right) = \int_{\tau}^{0} e^{-A_4'(\tau-\delta)_1} \bar{W}_3 e^{-A_4\tau-\sigma} d\sigma, \tag{2.3.11}$$

Consider the equation

$$-\mu A_1'\delta_2 - \delta_2 A_4 - \bar{W}_2 = 0, \tag{2.3.12}$$

$$-A_4'\delta_3 - \delta_3 A_4 - \bar{W}_3 = 0, \tag{2.3.13}$$

It is easy to show that if the matrix A_4 stable, then the matrices

$$\delta_2 = \int_0^{\infty} e^{\mu\bar{A}_1's} W_2 e^{A_4 s} ds, \quad \delta_3 = \int_0^{\infty} e^{A_4's} W_3 e^{A_4 s} ds$$

are the unique solutions of the equations (2.3.12) and (2.3.13) respectively. If the solution (2.3.10), (2.3.11) at $\tau \to -\infty$ tend to solutions of the equations (2.3.12), (2.3.13), then the well-known theorem Tikhonov, we can say that the initial value $(\tilde{V}_2(0), \tilde{V}_3(0)) = (0,0)$ belongs to the region of influence of the rest point (δ_2 , δ_3).

This raises the question: which kind of conditions the functions $\tilde{V}_2, \tilde{V}_3$ at $\tau \to -\infty$ tend to solutions of the equations (2.3.12), (2.3.13)?

On this task a positive response given by the following theorem.

Theorem is given for (2.3.11) and (2.3.13).

Theorem 2.3.2. Let A_4 - stable matrix. Then $\tilde{V}_3(\tau) \to \delta_3$ at $\tau \to -\infty$, if and only if the equality

$$\int_{\tau}^{0} e^{A_4'\sigma} \overline{W}_3 e^{A_4\sigma} d\sigma = e^{A_4'\tau} \delta_3 e^{A_4\tau} - \delta_3, \tag{2.3.14}$$

where δ_3 - solution of the equation (2.3.13).

Proof. Let the equality (2.3.14) function $\tilde{V}_3$ is written in the form

$$\tilde{V}_3(\tau) = e^{-A_4'\tau} \int_{\tau}^{0} e^{A_4'\sigma} \overline{W}_3 e^{A_4'\sigma} d\sigma e^{-A_4\tau} \tag{2.3.15}$$

Considering (2.3.14) from (2.3.15) we obtain

$$e^{-A_4'\tau} \int_{\tau}^{0} e^{A_4'\tau} \overline{W}_3 e^{A_4\tau} \cdot d\sigma \cdot e^{-A_4\tau} = \delta_3 - e^{-A_4'\tau} \delta_3 e^{-A_4\tau} \tau.. \tag{2.3.16}$$

Since by hypothesis of theorem the matrix A_4 is stable and from this follows that for $\tau \to -\infty$ $\tilde{V}_3(\tau) \to \delta_3$.

Suppose now, on the contrary: $\tilde{V}_3(\tau) \to \delta_3$ at $\tau \to -\infty$, where δ_3 – solution of equation (2.3.13). If so, then the integral (2.3.14) can be represented in the form

$$e^{-A_4'\tau}\int_{\tau}^{0} e^{A_4'\tau}\overline{W}_3 e^{A_4\tau}\cdot d\sigma\cdot e^{-A_4\tau}=\delta_3-e^{-A_4'\tau}\delta_3 e^{-A_4\tau}\tau. \qquad (2.3.16)$$

From this follows the equation (2.3.14). Now we show the validity of the equality (2.3.14) that δ_3 - solution of the equation (2.3.13).

Differentiating both sides of (2.3.14) by τ we have:

$$-e^{A_4'\tau}\overline{W}_3 e^{A_4\tau}=A_4' e^{A_4\tau}\delta_3 e^{A_4\tau}+e^{-A_4'\tau}\delta_3 e^{A_4\tau}A_4.$$

Multiplying this equality on the left by the matrix $e^{-A'\tau}$, right to matrix $e^{-A_4\tau}$, obtain the equivalent equation:

$$-\overline{W}_3=e^{-A_4'\tau}A_4 e^{A_4'\tau}\delta_3+\delta_3 e^{A_4\tau}A_4 e^{-A_4\tau}.$$

Considering the property of the matrix exponential for constant matrix A_4: $e^{A_4\tau}A_4=A_4e^{A_4\tau}$, we have from the last

$$-A_4'\delta_3-\delta_3A_4-\overline{W}_3=0.$$

By assumption δ_3 is a solution of (2.3.13), and therefore is obtained the identity.

The above theorem is valid for (2.3.10) (2.3.12).

Thus, the estimate of the integral reduces to the solution of algebraic equations (2.3.12), (2.3.13) in the semi-infinite interval $(0,\infty)$. Following Tikhonov's theorem, we arrive at the following conclusion:

If a) the matrices $A_i(t)$ $\left(i=\overline{1,4}\right)$ uniformly bounded and uniformly continuous together with its derivatives at $t\in[t_0,t_1]$;

b) $A_4(t)$ - stable matrix at $t \in [t_0, t_1]$, then exists a number μ_0 such that when $0 < \mu < \mu_0$ the solution of (2.3.14) exists and is unique in the segment $t_0 \le t \le t_1$.

The solution of problems (2.3.7a), (2.3.8), (2.3.9) can serve as the asymptotic behavior of solutions of (2.3.6) and when assessing the value of the integral (2.3.1) provide more accurate results than solutions problems (2.3.7).

Chapter 3

Method of Moments in the Theory of Singularly Perturbed Systems

After applying the method of moments to solve the problem of optimal control of linear systems with lumped parameters by Krasovsky N. N. [86] this method has been successfully compiled and turned out to be a very strong unit study many tasks of control. In this chapter will set out how to use the theory of moments to the problem of control of singularly perturbed systems with a variety of optimizable functionals.

3.1 Statement of the Problem on How to Manage the Problem of Moments

Let the behavior of the controlled system described by the equations

$$\begin{aligned} \dot{x} &= A_1(t)x + A_2(t)z + B_1(t)u + f_1(t), \quad x(t_0) = x^0, \\ \mu\dot{z} &= A_3(t)x + A_4(t)z + B_2(t)u + f_2(t), \quad z(t_0) = z^0, \end{aligned} \tag{3.1.1}$$

where $x(t) \in R^n$, $z(t) \in R^m$ - vectors of state, $u(t) \in R^r$ - control, $f_1(t) \in R^n$, $f_2(t) \in R^m$ - constantly operating outside forces; $t \in [t_0, t_1]$, μ - "small" positive parameter $(0 < \mu << 1)$.

It is assumed that the system

$$\dot{\bar{x}} = A_0(t)\bar{x} + B_0(t)u + f_0(t) \tag{3.1.2}$$

where $A_0(t) = A_1(t) - A_2(t) \cdot A_4^{-1}(t)A_3(t), \quad B_0(t) = B_1(t) - A_2(t) \cdot A_4^{-1}(t)B_2(t),$

$$f_0(t) = f_1(t) - A_2(t) \cdot A_4^{-1}(t) f_2(t)$$

is completely controllable and

$$\operatorname{Re} \lambda(A_4(t)) < 0\,. \tag{3.1.3}$$

It should be noted that some problems of control chosen value which characterizes the costs of resources for the implementation of the process control. Usually is required to achieve the desired result so that the value of this quantity was minimal and this value does not exceed certain limits. This value is called the criterion of optimality or intensity [86] control and denote it by the symbol $J(u)$.

Let any normed space of functions by symbol $M\{\cdot\}$ and will use it whenever the norm of the space of functions is not fixed. The symbol R_μ we will denote the perturbed problem, a R_0 - unperturbed problem of optimal control and at the same time J_μ^*, J_0^* – the minimum values of the criterion of optimality in problems R_μ^* and R_0 respectively.

We formulate the following problem R_μ : Let was selected criterion of optimality $J_\mu(u_\mu)$, which can be interpreted as a norm $\rho_\mu^*(u_\mu)$ functions $u_\mu(t) = u(t,\mu)$ in space $M^*\{u_\mu\}$.

It requires among admissible controls [86] to find the optimal control $u_\mu^0(t)$, which puts the system (3.1.1) from the initial state $x(t_0,\mu) = x^0$, $z(t_0,\mu) = z^0$ to the final state $x(t_1,\mu) = x^1$, $z(t_1,\mu) = z^1$ and thus having the smallest possible form $\rho_\mu(u_\mu^0)$.

This problem with the $\mu = 0$ responsible task of smaller dimension R_0 :

$$J_0(\overline{u}) \to \min_{\overline{u} \in M^0},$$

$$\overline{x} = A_0(t)\overline{x} + B_0(t)\overline{u} + f_0(t), \overline{x}(t_0) = x^0,$$

$$\overline{z} = -A_4^{-1}(t)[A_3(t)\overline{x} + B_2(t)\overline{u} + f_2(t)],$$

where $f_0(t) = f_1(t) - A_2(t) \cdot A_4^{-1}(t) f_2(t)$.

As shown in Chapter 1, the system (3.1.1) may be replaced system with separate movements

$$\begin{aligned} &\dot{\tilde{x}} = \tilde{A}_1(t,\mu)\tilde{x} + \tilde{B}_1(t,\mu)u + \tilde{f}_1(t,\mu),\ \tilde{x}(t_0) = \tilde{x}, \\ &\mu\dot{\tilde{z}} = \tilde{A}_4(t,\mu)\tilde{z} + \tilde{B}_2(t,\mu)u + \tilde{f}_2(t,\mu),\ \tilde{z}(t_0) = \tilde{z}^0, \end{aligned} \tag{3.1.4}$$

where

$$\tilde{A}_1(t) = \tilde{A}_1(t,\mu) = A_1(t) + A_2(t)H(t,\mu),\ \tilde{A}_4(t) = \tilde{A}_4(t,\mu) = A_4(t) - \mu H(t,\mu)A_2(t),$$

$$\tilde{B}_1(t) = \tilde{B}_1(t,\mu) = B_1(t) + N(t,\mu)\tilde{B}_2(t,\mu),\ \tilde{B}_2(t) = \tilde{B}_2(t,\mu) = B_2(t) - \mu H(t,\mu)B_1(t), \tag{3.1.5}$$

$$\tilde{f}_1(t) = \tilde{f}_1(t,\mu) = f_1(t) - N(t,\mu)\tilde{f}_2(t,\mu),\ \tilde{f}_2(t) = \tilde{f}_2(t,\mu) = f_2(t) - \mu H(t,\mu)f_2(t),$$

$\tilde{x}^0 = \tilde{x}^0(\mu) = x^0 - \mu N(t_0,\mu)\tilde{z}^0,\quad \tilde{z}^0 = \tilde{z}^0(\mu) = z^0 - H(t_0,\mu)x^0,\quad x^0, z^0$ - given vectors.

For small values of the parameter μ , matrices $H(t) = H(t,\mu),$ $N(t) = N(t,\mu)$ is having a dimension $m \times n,$ $n \times m$ are the solutions of singularly perturbed equations (see 1.1)

$$\mu\dot{H} + \mu H\tilde{A}_1(t) = A_3(t) + A_4(t)H,\ \ H(t_0) = H^0, \tag{3.1.6}$$

$$\mu\dot{N} - \mu\tilde{A}_1(t)N = -A_2(t) - N\tilde{A}_4(t),\ \ N(t_1) = N^1, \tag{3.1.7}$$

and $H(t,\mu) \to -A_4^{-1}(t)A_3(t),\ \ N(t,\mu) \to -A_2(t)A_4^{-1}(t),$ at $\mu \to 0$.

Let $\Phi(t,s,\mu)$ and $\Psi(t,s,\mu)$ are normalized at the point $t = s \quad (t,s \in [t_0,t_1])$ transition matrices of homogeneous systems $\dot{\tilde{x}} = \tilde{A}_1(t)\tilde{x},\ \ \mu\dot{\tilde{z}} = \tilde{A}_4(t)\tilde{z}.$

We write the law of motion of the system (3.1.4) by the Cauchy formula:

$$\tilde{x}(t.\mu) = \Phi(t,t_0,\mu)\tilde{x}^0 + \int_{t_0}^{t} \Phi(t,s,\mu)[\tilde{B}_1(s,\mu)u(s,\mu) + \tilde{f}_1(s,\mu)]ds, \tag{3.1.8}$$

$$\tilde{z}(t.\mu) = \Psi(t,t_0,\mu)\tilde{z}^0 + \frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)[\tilde{B}_2(s,\mu)u(s,\mu) + \tilde{f}_2(s,\mu)]ds. \quad (3.1.9)$$

In view of (3.1.5) for the functions (3.1.8), (3.1.9) can be easily checked by limiting relations $\lim_{\mu\to 0}\tilde{x}(t.\mu) = \bar{x}(t)$, $\lim_{\mu\to 0}\tilde{z}(t.\mu) = -A_4^{-1}(t)\left(B_2(t)\bar{u}(t) + f_2(t)\right)$, where $\bar{x}(t)$ - the vector of state "slow" subsystem (3.1.2). We introduce the following notation:

$$h_1(t,s,\mu) = \Phi(t,s,\mu)\tilde{B}_1(s,\mu), \;\; h_2(t,s,\mu) = \Psi(t,s,\mu)\tilde{B}_2(s,\mu). \quad (3.1.10)$$

Definition. Matrices $h_1(t,s,\mu)$ and $h_2(t,s,\mu)$, each line item form $r-$ dimensional vectors $h_{1i}(t,s,\mu),\;\; h_{2j}(t,s,\mu)$ will be called the "slow and" fast "pulse transition matrices of the system (3.1.4) on the impact of $u_\mu(t) = u(t,\mu)$.

Comment. In the future, we will assume within the meaning of the impulse response matrix [86] that $h^{(1)}(t,s,\mu)$=0, $h_2(t,s,\mu)$=0 at $t < s$.

In view of (3.1.10) relations (3.1.8) and (3.1.9) can be written in the form

$$\tilde{x}(t,\mu) = \Phi(t,t_0\mu)\,x^0 + \int_{t_0}^{t}\Phi(t,s,\mu)\;\tilde{f}_1(s,\mu)ds + \int_{t_0}^{t}h^{(1)}(t,s,\mu)u(s,\mu)ds \quad (3.1.11)$$

$$\tilde{z}(t,\mu) = \Psi(t,t_0\mu)\,\tilde{z}^0 + \frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\;\tilde{f}_2(s,\mu)ds + \frac{1}{\mu}\int_{t_0}^{t}h^{(2)}(t,s,\mu)u(s,\mu)ds. \quad (3.1.12)$$

Substituting the boundary conditions $\tilde{x}(t_1,\mu) = \tilde{x}^1, \tilde{z}(t_1,\mu) = \tilde{z}^1$ in (3.1.10), (3.1.12), integral equations

$$\int_{t_0}^{t_1} h_{1i}(t_1,s,\mu)u(s,\mu)ds = \alpha_{1i}(\mu), \quad i=\overline{1,n} \tag{3.1.13}$$

$$\int_{t_0}^{t_1} h_{2j}(t_1,s,\mu)u(s,\mu)ds = \mu\alpha_{2j}(\mu), \ j=\overline{1,m} \tag{3.1.14}$$

where $\alpha_{1i}(\mu) = \tilde{x}_i^1 - \phi^{[i]'}(t_1,t_0,\mu)\,\tilde{x}_i^0 - \int_{t_0}^{t_1} \phi^{[i]'}(t_1,s,\mu)\,\tilde{f}_1(s,\mu)ds,$

$$\alpha_{2j}(\mu) = \tilde{z}_j^1 - \psi^{[j]'}(t_1,t_0,\mu)\,\tilde{z}_j^1 - \frac{1}{\mu}\int_{t_0}^{t_1} \psi^{[j]'}(t_1,s,\mu)\,\tilde{f}_2(s,\mu)ds,$$

$$x_i^1 = x_i^1(\mu) = x_i^1 - \mu N^{[i]'}(t_1,\mu)\tilde{z}_j^1, \ \ z_i^1 = z_i^1(\mu) = z_i^1 - H^{[i]'}(t_1,\mu)x_i^1,$$

x_i^1, z_j^1 - given numbers, $\phi^{[i]'}(t_1,t_0,\mu)$, $\psi^{[j]'}(t_1,t_0,\mu)$, $N^{[i]'}(t_1,\mu), H^{[j]'}(t_1,\mu)$ - vectors - lines whose components are formed from elements of the rows of the respective matrix $\Phi(t_1,t_0,\mu)$, $\Psi(t_1,t_0,\mu)$, $N(t_1,\mu)$, $H(t_1,\mu)$.

In accordance with the wording of the problem R_μ vector function $h_{1i}(t_1,t,\mu)$, $h_{2j}(t_{1,}\,t,\mu)$ $\left(i=\overline{1,n}; j=\overline{1,m}\right)$ can be considered as elements of a space $M\left\{h_\mu(t)\right\}$, and the vector function $u_\mu(t)$ depicting the control as elements of the space $M^*\left\{u_\mu(t)\right\}$ adjoint to the $M\left\{h_\mu(t)\right\}$.

Then the problem R_μ is reduced to the problem of moments [42]. Left side of (3.1.13) and (3.1.14) are the linear operation $g\left[h_\mu(t)\right]$ performed on the elements $h_{\mu i}^{(1)}(t) = h_{1i}(t_1,t,\mu)$ $\left(i=\overline{1,n}\right)$, $h_{\mu j}^{(2)}(t) = h_{2j}(t_1,t,\mu)$ $\left(j=\overline{1,m}\right)$.

We formulate the problem of moments for the task R_μ :

Is required to find the linear operation $g\left[h_\mu(t)\right]$ certain space $M\left\{h_\mu(t)\right\}$, satisfying at predetermined elements $h_{\mu i}^{(1)}(t)$ $\left(i=\overline{1,n}\right)$, $h_{\mu j}^{(2)}(t)$ $\left(j=\overline{1,m}\right)$ conditions

$$g\left[h_{\mu i}^{(1)}(t)\right]=\alpha_{1i},\ \ i=\overline{1,n}; \qquad (3.1.15)$$

$$g\left[h_{\mu j}^{(2)}(t)\right]=\mu\alpha_{2j},\ \ j=\overline{1,m}$$

and at the same norm $\rho_\mu^*[g]$, operations $g\left[h_\mu(t)\right]$, was the lowest of the possible.

Each linear operation that makes sense for functions $h_\mu(t)$, from $M\left\{h_\mu(t)\right\}$ is generated by a control $u_\mu(t)$, in integral form [42]. Therefore, interpreting the expression on the left side of (3.1.13) and (3.1.14) as a linear function of the operation generated $u_\mu(t)=u(t,\mu)$ can be replaced by the problem of determining $u_\mu(t)$ problem of moments, i.e. the task of determining the operation $g\left[h_\mu(t)\right]$ satisfying (3.1.15).

Then, in this case, according to the problem of moment [86], we need to find from family vector of function of the form

$$h(t_1,t,\mu)=l_1'h_1(t_1,t,\mu)+l_2'h_2(t_1,t,\mu) \qquad (3.1.16)$$

function $h^0(t_1,t,\mu)$, at which the minimum

$$\begin{aligned}\rho_\mu^0&=\min_{l_1,l_2}\rho[l_1'h_\mu^{(1)}(t)+l_2'h_\mu^{(2)}(t)]\\&=\rho_\mu\left[l_1^{0\prime}h_\mu^{(1)}(t)+l_2^{0\prime}h_\mu^{(2)}(t)\right]=\rho_\mu^0[h_\mu^0(t)]\end{aligned} \qquad (3.1.17)$$

at $l_1'\alpha_1 + \mu l_2'\alpha_2 = 1,$

where

$l_1' = \left(l_{11}, l_{12}, \ldots, l_{1n}\right), \quad l_2' = \left(l_{21}, l_{22}, \ldots, l_{2m}\right), \quad h_\mu^{(1)}(t) = h_1\left(t_1, t, \mu\right), \quad h_\mu^{(2)}(t) = h_2\left(t_{1,} t, \mu\right),$

$\rho_\mu[h_\mu(t)]$ - norm of a function in the space $\mathrm{M}\left\{h_\mu(t)\right\}$.

Function $h^0\left(t_1, t, \mu\right)$ we call the minimum function.

Minimum function $\bar{h}_0^0\left(t_1 t\right)$ for the task R_0 is determined from the condition

$$\rho_0^0 = \min_{l_1}[l_1'\bar{h}_0(t)] = \rho_0[l_1'^{0\prime}\bar{h}_0(t)] = \rho_0^0[\bar{h}^0(t)] \tag{3.1.18}$$

at $l_1'\bar{\alpha}_1 = 1,$

where $\bar{h}_0(t) = \bar{h}_0\left(t_1, t\right) = \bar{\Phi}_0\left(t_1, t\right)\mathrm{B}_0(t), \quad \bar{\alpha}_1 = x^1 - \bar{\Phi}_0\left(t_1, t\right)x^0 - \int_{t_0}^{t_1} \bar{\Phi}_0\left(t_1, \tau\right) f_0(\tau)\, d\tau,$

$$h^0(t) = l^{0\prime}\bar{h}_0(t).$$

3.2 Control with Minimal Power

Consider the following problem of optimal control: it is required to find a control

$$u = u_\beta^0(t, \mu) \quad (u_\beta^0 \in R^1)$$

transforming system

$$\begin{aligned} \dot{x} &= A_1(t)x + A_2(t)z + B_1(t)u + f_1(t) \\ \mu\dot{z} &= A_3(t)x + A_4(t)z + B_2(t)u + f_2(t) \end{aligned} \tag{3.2.1}$$

from the initial state

$$x(t_0) = x^0, \quad z(t_0) = z^0 \tag{3.2.2}$$

to the final state

$$x(t_1) = x^1, \quad z(t_1) = z^1 \tag{3.2.3}$$

provided that the norm of

$$\|u\|_{L_\infty} = \max_{t_0 \le t \le t_1} |u(t, \mu)| \tag{3.2.4}$$

has reached the minimum value. Here $x \in R^n$, $z \in R^m, u \in R^1$, μ - small parameter.

Suppose that for the problem (3.2.1) - (3.2.4) the following conditions:

1^0. Matrices $A_i(t)$ $(i = \overline{1,4})$, $B_j(j = \overline{1,2})$ - uniformly bounded and uniformly continuous together with its derivatives at $t \in [t_0, t_1]$; matrix $A_4(t)$ nondegenerate, i.e. exist $A_4^{-1}(t)$.

2^0. Vectors $L_1(t), L_2(t), \ldots, L_n(t)$ are linearly independent, at least at one $t^* \in (t_0, t_1)$, i.e. $\sum_{i=1}^{n} v_i L_i(t^*) \neq 0$ at $\sum_{i=1}^{n} v_i^2 \neq 0$, where $L_1(t) = B_0(t) = B_1(t) - A_2(t) A_4^{-1}(t) B_2(t)$,

$$L_k(t) = A_0(t) L_{k-1} - \frac{dL_{k-1}}{dt}, \quad A_0(t) = A_1(t) - A_2(t) A_4^{-1}(t) A_3(t), \quad k = 2, 3, \ldots, n;$$

3^0. At point $t = t_1$

$$rank\{B_2(t_1), A_4(t_1) B_2(t_1), \ldots, A_4^{m-1}(t_1) B_2(t_1)\} = m; \tag{3.2.5}$$

4^0. Roots $\lambda_i(t)$ of characteristic equation of the matrix $A_4(t)$ subject to inequality

$$\operatorname{Re}\lambda_i(t) \le -\gamma < 0 \quad (i=\overline{1,m}; \quad t\in[t_0,t_1]); \tag{3.2.6}$$

In case, when $A_i(t) \quad (i=\overline{1,4}), \quad B_j(j=\overline{1,2})$ - constant matrices, instead of the condition 1^0, 2^0 we make the following demands:

5^0. $rank\{B_0, A_0B_0, \ldots, A_0^{n-1}B_0\} = n;$ (3.2.7)

6^0. $rank\{B_2, A_4B_2, \ldots, A_4^{m-1}B_2\} = m.$ (3.2.8)

When the condition 1^0, 4^0 in the system (3.2.1) can make a complete separation of movements. After simple transformations (See Chapter 1), we obtain:

$$\dot{\tilde{x}} = \tilde{A}_1(t,\mu)\tilde{x} + \tilde{B}_1(t,\mu)u + \tilde{f}_1(t,\mu), \tag{3.2.9}$$

$$\mu\dot{\tilde{z}} = \tilde{A}_4(t,\mu)\tilde{z} + \tilde{B}_2(t,\mu)u + \tilde{f}_2(t,\mu),$$

$$\tilde{x}(t_0) = \tilde{x}^0, \quad \tilde{z}(t_0) = \tilde{z}^0, \tag{3.2.10}$$

$$\tilde{x}(t_1) = \tilde{x}^1, \quad \tilde{z}(t_1) = \tilde{z}^1, \tag{3.2.11}$$

where $\tilde{A}_1(t,\mu) = A_1(t) - A_2(t)H(t,\mu), \quad \tilde{A}_4(t,\mu) = A_4(t) - \mu H(t,\mu)A_2(t),$

$$B_1(t,\mu) = B_1(t) + N(t,\mu)\tilde{B}_2(t,), \quad \tilde{B}_2(t,\mu) = B_2 - \mu H(t,\mu)B_1(t),$$

$$\tilde{f}_1(t,\mu) = f_1(t) - N(t,\mu)\tilde{f}_2(t,\mu), \quad \tilde{f}_2(t,\mu) = f_2(t) - \mu H(t,\mu)f_1(t),$$

$$\tilde{x} = x + \mu H(t,\mu)\tilde{z}, \qquad \tilde{z} = z - H(t,\mu)x,$$

$$\tilde{x}^\nu = x^\nu + \mu H(t_\nu,\mu)\tilde{z}^\nu, \qquad \tilde{z}^\nu = z^\nu - H(t_\nu,\mu)x^\nu, \quad \nu = 0,1\ldots;$$

Matrices $N(t,\mu)$ and $H(t,\mu)$ are solutions of the equations

$$\mu\dot{N} - \mu\tilde{A}_1N = -A_2 - N\tilde{A}_4, \tag{3.2.12}$$

$$\mu\dot{H} + \mu H\tilde{A}_1 = A_3 + \tilde{A}_4H. \tag{3.2.13}$$

Matrices elements N, H regular dependent parameters. As shown in Chapter 1, the matrices N and H of (3.2.12), (3.2.13) are uniquely determined, whereby instead of the problem (3.2.1) - (3.2.4) to consider the problem (3.2.9) - (3.2.11), (3.2.4). This problem reduces to the problem of moments: to find

$$\rho_\beta^0 = \min_{p,q} \int_{t_0}^{t_1} \left| \tilde{B}_1'(\sigma,\mu)\Phi'(t_1,\sigma,\mu)\,p + \tilde{B}_2'(\sigma,\mu)\Psi'(t_1,\sigma,\mu)\,q \right| d\sigma, \qquad (3.2.14)$$

on condition

$$C_1' p + \mu C_2' q = 1, \qquad (3.2.15)$$

where $\Phi(t,s,\mu)$ and $\Psi^{'}(t,s,\mu)$ $(t_0 \le t \le t_1)$ - normalized at the point $s \in [t_0,t_1]$ the fundamental matrices of homogeneous systems $\dot{\tilde{x}} = \tilde{A}_1 x, \quad \mu\dot{\tilde{z}} = A_4\tilde{z}$ respectively;

$$C_1 = \tilde{x}^1 - \Phi(t_1,t_0,\mu)\tilde{x}^0, \quad C_2 = \tilde{z}^1 - \Psi(t_1,t_0,\mu)\tilde{z}^0. \qquad (3.2.15a)$$

For the matrix $\Psi(t,s,\mu)$ the condition [42]

$$\left\|\Psi(t,s,\mu)\right\| \le C\exp\left(-\gamma_1(t-s)\big/\mu\right)$$

for all $t,s \in [t_0,t_1], \quad t \ge s,$ when $C,\gamma_1 > 0 - const.$

If we assume that in some way solved the problem (3.2.14), (3.2.15) and thus found vectors $p = p^0, q = q^0,$ then it will be known the minimum function

$$h_\beta^0(t,\mu) = \tilde{B}_1'(t,\mu)\Phi'(t_1,t,\mu)\,p^0 + \tilde{B}_2'(t,\mu)\Psi'(t_1,t,\mu)q^0 \qquad (3.2.16)$$

and the number of $p_\beta^0 > 0.$

According to the rule the problem of the moment [86], we have to define the desired optimal control $u = u_\beta^0(t,\mu)$ based on the maximum condition

$$\int_{t_0}^{t_1} h_\beta^0(t,\mu)u_\beta^0(t,\mu)dt = \max_u \int_{t_0}^{t_1} h_\beta^0(t,\mu)u(t,\mu)dt = 1 \qquad (3.2.17)$$

at $\max\limits_{t_0 \le t \le t_1} |u(t,\mu)| = \dfrac{1}{\rho_\beta^0}$ or else when $|u(t,\mu)| = \dfrac{1}{\rho_\beta^0}$.

Maximum integral in (3.2.17) will be achieved if each time t integrand $h^0(t,\mu)\,u(t,\mu)$ will be maximum. Thus, optimal control $u_\beta^0(t,\mu)$ must be determined from the condition [42]

$$h_\beta^0(t,\mu)u_\beta^0(t,\mu) = \max_u h_\beta^0(t,\mu)u(t,\mu) \qquad (3.2.18)$$

$$\text{at } |u(t,\mu)| \le \frac{1}{\rho_\beta^0} = \omega_\beta^0 \quad (t_0 \le t \le t_1). \qquad (3.2.19)$$

Since the system (1) non-singular, and hence the function $h_\beta^0(t,\mu)$ - a smooth [86], i.e. it is at a given time interval is zero only in a finite number of isolated values $t = t_j$. Then the solution of problem (3.2.18), (3.2.19) delivered by the expression

$$u_\beta^0(t,\mu) = \omega_\beta^0 sign(\tilde{B}_1'(t,\mu)\Phi'(t_1,t,\mu)p^0 + \tilde{B}_2'(t,\mu)\Psi'(t_1,t,\mu)q^0 \qquad (3.2.20)$$

$$(t_0 \le t \le t_1)$$

Function $u_\beta^0(t,\mu)$ is defined everywhere except for a finite number of isolated values $t = t_j$, where the function standing under the sign of «sing» vanishes.

At $\mu = 0$ from (3.2.9) - (3.2.11) we obtain

$$\dot{\bar{x}} = A_0(t)\bar{x} + B_0(t)\bar{u} + f_0(t), \quad \bar{x} = (t_\nu) = x^\nu, \quad \nu = 0;1\ldots; \qquad (3.2.21)$$

$$\bar{z} = -A_4^{-1}(t)\left(A_3(t)\bar{x} + B_2(t)\bar{u} + f_2(t)\right) \tag{3.2.22}$$

where $A_0 = A_1 - A_2A_4^{-1}A_3, \quad B_0 = B_1 - A_2A_4^{-1}B_2, \quad f_0(t) = f_1 - A_2A_4^{-1}f_2.$

The resulting system is called a generating system [26]. It should be noted that the solution of (3.2.21), (3.2.22), (3.2.4) can not serve as a zero approximation of the problem (3.2.9) - (3.2.11), (3.2.4) as the in the vicinity of the boundary interval $[t_0, t_1]$ there may be finite number of isolated points (to m pieces), which must go through the process of switching control action. Also in this case, the issue of system switching from one state to another for rapid subsystem (3.2.9) remains open. So first of all we need to specify a system for which the optimal solution of the problem is well-defined zero approximation of the problem (3.2.9) - (3.2.11), (3.2.4).

Due to requirements regarding $A_i(t) \quad (i = \overline{1,4})$ system (3.2.1) (see the condition 1^0) solutions of equations (3.2.12) - (3.2.13) are limited and at $\mu \to 0$ will be performed:

$$\tilde{A}_1(t,\mu) \to A_0(t), \quad \tilde{A}_4(t,\mu) \to A_4(t), \quad \tilde{B}_1(t,\mu) \to B_0(t), \quad \tilde{B}_2(t,\mu) \to B_2(t),$$
$$\tilde{f}_1(t,\mu) \to f_0(t), \quad \tilde{f}_2(t,\mu) \to f_2(t).$$

Consider the system

$$\dot{\bar{x}} = A_0(t)\bar{x} + B_0(t)\bar{u} + f_0(t), \quad \bar{x} = (t_\nu) = x^\nu \tag{3.2.23}$$

$$\mu\dot{\bar{x}}_* = A_4(t_1)\bar{z}_* + B_2(t_1)\bar{u} + f_2(t), \quad \bar{z}_* = (t_\nu) = z_*^{\ \nu} \tag{3.2.24}$$

where $z_* = \bar{z} + A_4^{-1}(t_1)A_3(t_1)\bar{x}, \quad z^\nu{}_* = z^\nu + A_4^{-1}(t_1)A_3(t_1)x^\nu, \quad \nu = 0;1\ldots$

The system (3.2.23), (3.2.24) approximates the system (3.2.9) - (3.2.11) with accuracy of the order of smallness $O(\mu)$ and it is obtained from (3.2.9) - (3.2.11) in the following approximations:

$$H(t,\mu)\approx H_0(t)=-A_4^{-1}(t)A_3(t),\quad N(t,\mu)\approx N_0(t)=-A_2(t)A_4^{-1}(t),\quad \tilde{A}_4(t_1+\tau\mu)\approx A_4(t_1),$$

$$\tilde{B}_2(t_1+\tau\mu)\approx B_2(t_1),\quad -\infty<\tau\le 0.$$

For the new system the minimum function takes the form

$$h_0^0(t,\mu)=h_0^0(t,\overline{p}^0,\overline{q}^0,\mu)=B_0'(t)\overline{\Phi}'(t_1,t)\overline{p}^0+B_2'(t_1)e^{-A_4'(t_1)\frac{t-t_1}{\mu}}\overline{q}^0, \qquad (3.2.25)$$

where $\overline{\Phi}'(t_1,s)$ - the fundamental matrix of the homogeneous system $\dot{\overline{x}}=A_0\overline{x};\quad \overline{p}^0,\overline{q}^0$ - solutions extremal problem: find

$$\rho_0^0=\min_{\overline{p},\overline{q}}\int_{t_0}^{t_1}\left|h_0(t,\overline{p},\overline{q},\mu)\right|dt \qquad (3.2.26)$$

on condition

$$\overline{C}_1'\overline{p}+\overline{C}_2'\overline{q}=1 \qquad (3.2.27)$$

$$\overline{C}_1=x'-\overline{\Phi}(t_1,t_0)x^0,\quad \overline{C}_2=z_*^1-e^{-A_4(t_1)\tau_0}z_*^0=z_*^1+O(e^{-\gamma\tau_0})\approx z_*^1,\quad \tau_0=\frac{t_0-t_1}{\mu}. \qquad (3.2.27a)$$

Optimal control (3.2.20) for this case is written as

$$u_0^0(t,\mu)=\omega_0^0 sign\left(\tilde{B}_0'(t,\mu)\Phi'(t_1,t)\overline{p}^0+\tilde{B}_2'(t_1)e^{-A_4(t_1)\tau}\overline{q}^0\right),$$
$$\tau=\frac{t-t_1}{\mu},\, t_0\le t\le t_1,\, \omega_0^0=\frac{1}{\rho_0^0}. \qquad (3.2.28)$$

Control (3.2.28) transfers the system (3.2.23), (3.2.24) from the initial states (x^0,z^0) to the final state (x^1,z^1) and it is the function of the relay. Minimum function $h_0^0(t,\mu)$ (3.2.25) may be zero in the vicinity of t_0,t_1, because it contains a function of boundary layer type, whereby the control (3.2.28) has a complete set of switching points, which is not always possible for the generating system (3.2.21), (3.2.22). Rewrite the equality (3.2.27) in the form of

$$\sum_{i=1}^{n}\bar{C}_i^{(1)}\bar{p}_1+\mu\sum_{k=1}^{m}\bar{C}_k^{(2)}\bar{q}k=1. \qquad (3.2.29)$$

We now show one of the approximate methods of determining the optimal parameters $\bar{p}_i^0,\bar{q}_k^0 \quad (i=\overline{1,n}; \quad k=\overline{1,m}).$

Assuming that the vector $\bar{C}_1$ (3.2.27) satisfies $\bar{C}_n^{(1)}\neq 0$, from (3.2.29), we obtain

$$\bar{p}_n=\frac{1}{\bar{C}_n^{(1)}}(1-\sum_{i=1}^{n-1}\bar{C}_i^{(1)}\bar{p}_i-\mu\sum_{k=1}^{m}\bar{C}_2^{(2)}\bar{q}_k). \qquad (3.2.30)$$

The following functions $h_0(t,\mu)=h_0(t,\bar{p},\bar{q},\mu)$ standing under the sign of the module in (3.2.26) can be represented as

$$B_0'(t)\bar{\Phi}'(t_1,t)p=K(t,t_1)\bar{p}=\sum_{i=1}^{n}K_i(t,t_1)\bar{p}_i, \qquad (3.2.31)$$

$$B_0'(t)\bar{\Phi}'(t_1,t)p=K(t,t_1)\bar{p}=\sum_{i=1}^{n}K_i(t,t_1)\bar{p}_i, \qquad (3.2.32)$$

where $\eta(\frac{t-t_1}{\mu})$ - function of the type of the boundary layer, in other words, for it has the estimate $\|\eta\|\leq C\exp\left(\gamma\left(\frac{t-t_1}{\mu}\right)\right), \quad C,\gamma>0-const$.

In view of (3.2.30) - (3.2.32) function $h_0(t,\mu)$ is written in the form

$$\begin{aligned}h_0(t,\mu)&=h_0(t,\tilde{p},\bar{q},\mu)\\&=\frac{K_n(t_1,t)}{C_n^{(1)}}+\left(\tilde{K}'(t,t_1)-\frac{K_n(t,t_1)}{C_n^{(1)}}\tilde{C}_1'\right)\tilde{p}+\left(\eta'(\frac{t-t_1}{\mu})-\mu\frac{K_n(t_1,t)}{C_n^{(1)}}\tilde{C}_2'\right)\bar{q},\end{aligned} \qquad (3.2.33)$$

$$\tilde{K}'=(K_1,K_2,\dots,K_{n-1}), \quad \tilde{p}'=(\bar{p}_1,\bar{p}_2,\dots,\bar{p}_{n-1}),$$

where $\eta' = (\eta_1, \eta_2, \ldots, \eta_m)$, $\quad \bar{C}_1' = (\bar{C}_1^{(1)}, \bar{C}_2^{(1)}, \ldots, \bar{C}_{n-1}^{(1)})$,

$$\tilde{C}_2' = (\bar{C}_1^{(1)}, \bar{C}_2^{(2)}, \ldots, \bar{C}_m^{(2)}), \quad \bar{q}' = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_m).$$

In this case, the problem (3.2.26), (3.2.27) the relative minimum is reduced to the problem of the absolute minimum of the function

$$\rho_0(\tilde{p}, \bar{q}, \mu) = \int_{t_0}^{t_1} \left| h_0(t, \tilde{p}, \bar{q}, \mu) \right| dt. \tag{3.2.34}$$

Note that for the functions $h_0(t, \tilde{p}, \bar{q}, \mu)$, $\rho_0(\tilde{p}, \bar{q}, \mu)$ at $\mu \to 0$ holds the following limit relations:

$$\lim_{\mu\to 0} h_0(t, \tilde{p}, \bar{q}, \mu) = \bar{h}_0(t, \tilde{p}), \quad \lim_{\mu \to 0} \rho_0(t, \tilde{p}, \bar{q}, \mu) = \bar{\rho}_0(\tilde{p}), \tag{3.2.35}$$

where $\bar{h}_0(t, \tilde{p}) = \dfrac{K_n(t_1, t)}{C_n^{(1)}} + \left(\tilde{K}'(t, t_1) - \dfrac{K_n(t, t_1)}{C_n^{(1)}} \tilde{C}_1' \right) \tilde{p}, \quad \rho_0(\tilde{p}) = \int_{t_0}^{t_1} \left| \bar{h}_0(t, \tilde{p}) \right| dt.$

Numbers $\bar{p}_i = \bar{p}_i^0$ $(i = \overline{1, n-1})$, $\bar{q}_k = \bar{q}_k^0$ $(k = \overline{1, m})$ determining the minimum function $h_0^0(t, \bar{p}^0, \bar{q}^0, \mu)$ will satisfy the system of equations

$$\begin{aligned} \frac{\partial \rho_0}{\partial \bar{p}_i} &= \int_{t_0}^{t_1} \left(K_i(t, t_1) \frac{K_n(t, t_1)}{C_n^{(1)}} \tilde{C}_1' \right) sign h_0(t, \tilde{p}, \bar{q}, \mu) dt = 0, \quad i = \overline{1, n-1}, \\ \frac{\partial \rho_0}{\partial \bar{q}_k} &= \int_{t_0}^{t_1} \left(\eta_i \left(\frac{t - t_1}{\mu} \right) \frac{\mu K_n(t, t_1)}{C_n^{(1)}} C_k^{(2)} \right) sign h_0(t, \tilde{p}, \bar{q}, \mu) dt = 0, \quad k = \overline{1, m}. \end{aligned} \tag{3.2.36}$$

As stated in [86] for the solution of this problem on the conditional minimum is considered the differential equations for the unknown parameters $l_k (k = \overline{1, M-1})$ (in this case, relatively $\bar{p}_i, \bar{q}_k$)

$$\frac{dl_i}{dv} = -\varepsilon \frac{\partial \rho(l_1, l_2, \ldots l_{M-1})}{\partial l_i}, \quad i = \overline{1, M-1} \tag{3.2.37}$$

where $\varepsilon > 0$ - coefficient of proportionality determining the "shutter" speed. In drawing up the differential equation (3.2.37) introduced a new parameter v, which is interpreted as the time counted at the point of the movement $l = \{l_i\}$ along "the descent of the curve" by some arbitrary point $\overline{l} = \{\overline{l_i}\}$ on the hyperplane $\sum_{l=1}^{M} C_i l_i = 1$ to the desired point $l^0 = \{l_i^0\}$, numerical integration by using the recurrence relation

$$l_i^{(j+1} = l_i^{(j)} - \varepsilon[\frac{\partial\rho(l_1, l_2, \ldots l_{M-1})}{\partial l_i}]_{l=l^{(j)}} \Delta v. \qquad (3.2.38)$$

From the second equation (3.2.36), we note that arbitrary $\frac{\partial\rho}{\partial\overline{q}_k} \quad (k = \overline{1,m})$ defined functions faster components $h_0(t, \tilde{p}, \overline{q}, \mu)$.

Then, in this case, will consider the following singularly perturbed differential equations respectively to $\overline{p}_i, \quad \overline{q}_k \quad \left(i = \overline{1, n-1}; \quad k = \overline{1,m}\right)$

$$\frac{d\overline{p}_i}{dv} = -\varepsilon\frac{\partial\rho_0}{\partial\overline{p}_i}, \quad \mu\frac{d\overline{q}_k}{dv} = -\varepsilon\frac{\partial\rho_0}{\partial\overline{q}_k}, \qquad (3.2.39)$$

where the partial derivatives $\frac{\partial\rho_0}{\partial\overline{p}_i}, \quad \frac{\partial\rho_0}{\partial\overline{q}_k}$ are defined by (3.2.36). For the numerical integration of the equation (3.2.39) can propose the following process of successive approximations. At $\mu = 0$ from (3.2.39) we obtain the reduced system

$$\frac{d\overline{p}_i}{dv} = -\varepsilon\frac{\partial\rho_0\left(\tilde{p}\right)}{\partial\overline{p}_i}, \quad \left(i = \overline{1, n-1}\right). \qquad (3.2.40)$$

Equation (3.2.40) can be integrated numerically using the relation (3.2.38), identifying all $p_i = p_i^0 \quad \left(i = \overline{1, n-1}\right)$ and leaving them in the equation

$$\mu \frac{d\overline{q}_k}{dv} = -\varepsilon \frac{\partial \rho_0 \left(\tilde{p}^0, \overline{q}\right)}{\partial \overline{q}_k}, \quad k = \overline{1, m} \tag{3.2.41}$$

and making the substitution $\tau = \dfrac{t - t_1}{\mu}$ the right side of (3.2.41), we obtain:

$$\frac{d\overline{q}_k}{dv} = -\varepsilon \int_{\tau_0}^{0} \left[\eta_k \left(\sigma, \mu\right) - \mu \frac{b_n^{(0)} C_k^{(2)}}{C_n^{(1)}} \right] sign \tilde{h}_0 \left(\sigma, \mu\right) d\sigma, \tag{3.2.42}$$

where

$$\begin{aligned} \tilde{h}(\sigma, \mu) &= h_0\left(\sigma\mu + t_1, \tilde{p}^0, \overline{q}, \mu\right) = h_0\left(t_1, \tilde{p}^0, \overline{q}\right) + h_0'\left(t_1, \tilde{p}, q\right)\sigma\mu + \ldots \approx \\ &\approx \sum_{i=1}^{n-1} \left[b_i^{(0)} - \frac{C_i^{(1)}}{C_n^{(1)}} b_n^{(0)} \right] \overline{p}_i^0 + \frac{1}{C_n^{(1)}} b_n^{(1)} + \sum_{k=1}^{m} \left[\eta_k\left(\sigma, \mu\right) - \mu \frac{b_n^{(0)} C_k^{(2)}}{C_k^{(1)}} \right] \overline{q}_k, \end{aligned} \tag{3.2.43}$$

where $\overline{p}^0 = \left(\overline{p}_1^0, p_2^0, \ldots, \overline{p}_{n-1}{}^0\right), \quad b_i^0 \left(i = \overline{1, n}\right)$ - vector components $B_0 = B_0\left(t_1\right)$..

Now, again using the relations (3.2.44) can be integrated into the equation (3.2.42).

After the necessary calculations will be known $\overline{q}_k{}^0 \left(k = \overline{1, m}\right)$. Number $\rho_0{}^0$ calculated using the formula (3.2.34). Number $\overline{\omega}_0{}^0 = \dfrac{1}{\rho_0{}^0}$ characterizes the amplitude of the control action.

Suppose now that for sufficiently small $\mu\left(0 < \mu < \mu_0\right)$ and the boundary points $\left(x^0, z^0\right)$, $\left(x^1, z^1\right)$ following conditions are met:

a) the angles of intersection of the graph of (3.2.25)

$$h_0{}^0(t,\mu)=h_0{}^0\left(t,\tilde{p}^0,\bar{q}^0,\mu\right)=B_0'(t)\bar{\Phi}_0'(t,t_1)\,\bar{p}^0+B_2'(t_1)e^{-A_4^1(t_1)(t-t_1)/\mu}\bar{q}^0 \quad (3.2.44)$$

to t axis nonzero;

b) Jacobian $\dfrac{\partial\left(\dfrac{\partial\rho_0}{\partial\tilde{p}},\dfrac{\partial\rho_0}{\partial\bar{q}}\right)}{\partial\left(\tilde{p},\bar{q}\right)}$ at $p=\bar{p}^0,\bar{q}=\bar{q}^0$ nonzero.

When the above conditions function $h_0(t,\mu)=h_0(t,\tilde{p},\bar{q},\mu)$ vanishes only for a finite number of isolated points in time $t=t_j(\bar{p},\bar{q},\mu)\left(j=\overline{1,s}\right)$, are defined as a function of the magnitude unambiguous $\bar{p}_i$, $\bar{q}_k$, $\left(i=\overline{1,n};\quad k=\overline{1,m}\right)$ and the partial derivatives $\dfrac{\partial t_j}{\partial\bar{p}_i}$, $\dfrac{\partial t_j}{\partial\bar{q}_k}$ at $\bar{p}_i=\bar{p}_i{}^0$, $\bar{q}_k=q_k{}^0$ $\left(i=\overline{1,n};k=\overline{1,m}\right)$, exist as it follows from the implicit function theorem.

Even under this condition from the same implicit function theorem implies that $\bar{p}_i{}^0\left(i=\overline{1,n}\right)$ are continuously differentiable functions on $C_i^{(1)}\left(i=\overline{1,n}\right)$, and $\bar{q}_k{}^0\left(k=\overline{1,m}\right)$ are continuously differentiable functions on $C_i^{(1)}$ $\left(i=\overline{1,n}\right)$, $C_k{}^{(2)}$ $\left(k=\overline{1,m}\right)$. Then for small changes $\Delta C_i^{(1)}$, $\Delta C_k{}^{(2)}$ will be small changes in variables $\bar{p}_i^0$, $\bar{q}_k{}^0$, where are the estimates:

$$\left|\Delta\bar{p}_i^0\right|\le r_1\left\|\Delta c_1\right\| \quad (3.2.45)$$

$$\left|\Delta\bar{q}_k{}^0\right|\le r_2\left(\left\|\Delta c_1\right\|+\left\|\Delta c_2\right\|\right) \quad (3.2.46)$$

$$\left|\Delta\omega_0{}^0\right|\le r_3\left(\left\|\Delta c_1\right\|+\left\|\Delta c_2\right\|\right) \quad (3.2.47)$$

where r_i - positive numbers.

Changing the minimum function $h_0^0(t,\mu)$ depends not only on changes in the values $\bar{p}_i^o$, $\bar{q}_k^0$, and members of unregistered matrix decomposition $\Phi(t,t_1\mu)$ and $\psi(t,t_1\mu)$. Imagine matrices $\Phi(t,s,\mu)$, $\psi(t,s,\mu)$ in the shape of:

$$\Phi(t,s,\mu) = \bar{\Phi}(t,s) + \mu\phi(t,s,\mu) \tag{3.2.48}$$

$$\psi(t,s,\mu) = e^{A_4(t)(t-s)/\mu} + \xi(t,s,\mu)\,. \tag{3.2.49}$$

It is easy to show that for sufficiently small values $\mu < \mu^0$ functions φ and ξ satisfy the inequalities

$$\|\phi(t,s,\mu)\| \le d_2 C^2 (e^{t-s} - 1) e^{-m(t-s)} \tag{3.2.50}$$

$$\|\xi(t,s,\mu)\| \le C(e^{d_1 C(t-s)} - 1)\, e^{-\frac{\gamma(t-s)}{\mu}} \tag{3.2.51}$$

where $m > 1, d_1, d_2, C -$const, $0 < \mu \le \mu^0, \quad \mu^0 = \min\left\{\frac{1}{d_2 C}, \frac{\gamma}{d_1 C}\right\}$.

Then, for any vectors p and q hyperplane of (3.2.27), in particular in $p = \bar{p}^0, q = \bar{q}^0$ function $h_\beta(t,p,q,\mu) \quad (t_0 \le t \le t_1)$ can be represented as

$$h_\beta(t,\bar{p}^0,\bar{q}^0,\mu) = h_0(t,\bar{p}^0,\bar{q}^0,\mu) + O(\mu + e^{\gamma\tau}), \tag{3.2.52}$$

where $\tau = \dfrac{t-t_1}{\mu} < 0, \gamma > 0 -$const.

This means that the function $h_\beta(t,\bar{p}^0,\bar{q}^0,\mu)$ has as many zeros as had $h_0(t,\bar{p}^0,\bar{q}^0,\mu)$. These zeros are placed with precision $O(\mu + e^{\gamma\tau})$ (the distance from the boundary point $t = t_1$ with precision $O(\mu)$), near the respective zeros

$h_0(t,\bar{p}^0,\bar{q}^0,\mu)$. In view of (3.2.48) and (3.2.49) from (3.2.15a), (3.2.27a), we obtain

$$\Delta C_1 = \mu(N(\underline{t_1,\mu})\tilde{z}^1 - \phi(t_1,t_0,\mu)x^0 \\ - \Phi_0(t_1,t_0)N(t_0,\mu)\tilde{z}^0 - \mu\phi(t_1,t_0,\mu)N(t_0,\mu)z^0, \tag{3.2.53}$$

$$\Delta C_2 = \mu(e^{-A_4(t_1)\tau_0}\cdot H_1(t_0,\mu)x^0 - H_1(t_1,\mu)x') - \xi(t_1,t_0,\mu)\tilde{z}^0 \\ \approx -\mu H_1(t_1,\mu)x^1, \tag{3.2.54}$$

where $H_1(t,\mu)$ - limited function, which appears from the relation:

$$H(t,\mu) = -A_4^{-1}(t)A_3(t) + \mu H_1(t,\mu),\ \tau_0 = \frac{t_0 - h}{\mu} \le 0.$$

Considering that the matrices N,H limited and functions φ,ξ satisfy (3.2.50), (3.2.51), from (3.2.45) - (3.2.47), we obtain:

$$\left|\Delta\bar{p}_i^0\ \right| \le m_1\mu,\ \left|\Delta\bar{q}_k^0\right| \le m_2\mu,\ \left|\Delta\omega_0^0\right| \le m_3\mu,\ \ m_1,m_2,m_3,\gamma > 0 - \text{const.} \tag{3.2.55}$$

These estimates suggest that for all t, except for a set Q values t, a measure which satisfy the inequality

$$\sigma(Q) \le m_4\mu\ . \tag{3.2.56}$$

Control $u_0^0(t,\mu)$ differs from the optimal control $u_\beta^0(t,\mu)$ the original problem with the accuracy $O(\mu)$, i.e. will be performed the following inequality:

$$\left|u_\beta^0 - u_0^0\right| = \left|\Delta u_0^0\right| \le m_5\mu,\ \ m_4,m_5 - \text{const.} \tag{3.2.57}$$

Hence we have the following conclusion:

Theorem 3.2.1. If the conditions a), b), then for sufficiently small values $\mu < \mu_0 \left(\mu_0 = \min\left\{ \frac{1}{d_2 C}, \frac{\gamma}{d_1 C} \right\} \right)$

1) optimal control $u_\beta(t,\mu)$ (3.2.20) can be approximated by a control $u_0^0(t,\mu)$ (3.2.28) with precision $O(\mu)$;

2) at $\mu \to 0$ both control - $u_{\beta,} u_0^0$ one tends to the same limit, i.e. $\lim\limits_{\mu\to 0} u_\beta(t,\mu) = \lim\limits_{\mu\to 0} u_0^0(t,\mu) = u^*(t).$

This item is the optimal solution for the problem of the generating system (3.2.21), the order of which is lower than (3.2.1). All these statements are true for all t, except for a set Q values, a measure which is of the order of smallness $O(\mu)$.

In conclusion, it should be noted that the above method is easily offended by vector control case.

Chapter 4

Research Tasks of Optimal Control of Dynamic Processes Economy

4.1 Decomposition of an Extreme Problem of Interbranch Balance

The dynamic model of interbranch balance describes by a system of singularly perturbed differential equations

$$\begin{aligned} \dot{x} &= \left(E_k - A_1(t)\right)x - A_2(t)z - w^{(1)}, \\ \mu\dot{z} &= -A_3(t)x + \left(E_{n-k} - A_4(t)\right)z - w^{(2)}, \end{aligned} \tag{4.1.1}$$

where E_k, E_{n-k} – identity matrix sizes of $k \times k$, $(n-k)\times(n-k)$ respectively; $w^{(1)}, w^{(2)}$ – vectors with dimensions of the final product k and $n-k$ respectively.

The formulation of an extreme problem for the system (4.1.1) is similar [44], leads to some complex restrictions that in solving the problem must take into account that creates certain difficulties and, ultimately, the inability to obtain effectively implemented algorithm. Therefore, it is necessary to replace the system (4.1.1) is equivalent to the system, which are separated by slow and fast position.

Consider the system

$$\begin{aligned} \dot{x} &= \left(E_k - A_0(t)\right)x - u^{(1)}, \\ \mu\dot{\tilde{z}} &= \left(E_{n-k} - A_4(t)\right)\tilde{z} - u^{(2)}, \end{aligned} \tag{4.1.2}$$

where

$$\begin{gathered} A_0 = A_1 - A_2\left(E_{n-k} - A_4\right)^{-1}A_3, \quad \tilde{z} = z + \left(E_{n-k} - A_4\right)^{-1}A_3 x, \qquad u^{(2)} = w^{(2)}, \\ u^{(1)} = w^{(1)} - A_2\left(E_{n-k} - A_4\right)^{-1}w^{(2)}. \end{gathered} \tag{4.1.3}$$

As shown in Section 1.2, such a system may change the original system, since it has all the properties of the original and is an oversimplification. Given the initial conditions and parameters known to control the decision of extreme problems for the system (4.1.2) can serve as an approximate solution of the extremal problem for the system (4.1.1) with precision $O(\mu)$.

We take as the control parameters the flow of final consumption w_j. The vector flow of consumption must obey certain natural limitations. The simplest kind of restrictions can be expressed by the following requirements:

1. In each branch the flow of consumption w_j can not be less than the specified minimum permissible value $\psi_j \geq 0$, i.e. $w_j \geq \psi_j, \quad j = 1, 2, ...n.$ Then naturally function $u^{(1)}(t), u^{(2)}(t)$ obtain following restrictions:

$$\begin{aligned} &u^{(1)} \geq \psi^{(1)} - A_2 \left(E_{n-k} - A_4\right)^{-1} \psi^{(2)}, \\ &u^{(2)} \geq \psi^{(2)}, \end{aligned} \tag{4.1.4}$$

where $u^{(1)}, \psi^{(1)} - k$ - dimensional, $u^{(2)}, \psi^{(2)} - (n-k)$ - dimensional vector functions.

The minimum allowable flow of consumption $\psi_j(t)$ can be determined by the rate consumption of production and of the population.

2. Vector flow accumulation looks: $\bar{s} = \begin{pmatrix} E_k & 0 \\ 0 & \mu E_{n-k} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \geq 0.$

In this case, the system (4.1.2), we have:

$$\psi^{(1)} - A_2 \left(E_{n-k} - A_4\right)^{-1} \psi^{(2)} \leq u^{(1)} \leq \left(E_k - A_0\right) x, \tag{4.1.5}$$

$$\psi^{(2)} \leq u^{(2)} \leq \left(E_{n-k} - A_4\right) \tilde{z}. \tag{4.1.6}$$

Thus, we must assume that the control functions subsystems (4.1.2) associated restrictions (4.1.5) and (4.1.6).

The optimality criterion, we can select a functional:

$$J = \int_{t_0}^{t_1} \left[\left(q^{(1)}, w^{(1)} \right) + \left(q^{(2)}, w^{(2)} \right) \right] dt \,. \tag{4.1.7}$$

where $\left(q^{(1)}, w^{(1)} \right) = \sum_{i=1}^{k} q_i w_i$, $\left(q^{(2)}, w^{(2)} \right) = \sum_{j=k+1}^{n} q_j w_j$, $q_i \geq 0$ $i = 1,2,...,n$ - a nonnegative decreasing function. The functional (4.1.7) represents the total amount of increase of social welfare for the period $\left[t_0, t_1 \right]$ [50]. Of course, we must strive to build such functions $w_j(t)$, $j = 1,2,...,n,$ that deliver the greatest value of this functionality.

Now, taking into account the relations (4.1.3) functional (4.1.7) may be represented as:

$$J = \int_{t_0}^{t_1} \left(q^{(1)}, u^{(1)} \right) dt + \int_{t_0}^{t_1} \left(\tilde{q}^{(2)}, u^{(2)} \right) dt = J^{(1)} + J^{(2)}, \tag{4.1.8}$$

where $\left(q^{(1)}, u^{(1)} \right) = \sum_{i=1}^{k} q_i^{(1)} u_i^{(1)}$, $\left(\tilde{q}^{(2)}, u^{(2)} \right) = \sum_{i=k+1}^{n} \tilde{q}_i^{(2)} u_i^{(2)}$, $\tilde{q}_i^{(2)} = \sum_{j=1}^{k} q_j^{(1)} r_{ji} + q_i^{(2)}$,

r_{ji} – elements of the matrix $R = A_2 \cdot \left(E_{n-k} - A_4 \right)^{-1}$.

It should be noted that the slow and fast sub system (4.1.2) are not connected, and it is possible to consider each separately. Furthermore, as the functional J represents the sum. The same function $u^{(1)} = u_0^{(1)}$ delivers the extreme values of the functionals J and $J^{(1)}$ with the following restrictions:

a) the rate of production growth $\dot{x} = \left(E_k - A_0(t) \right) x - u^{(1)}$,

b) on the initial and final output of production $x(t_0) = x^0, \qquad x(t_1) = x^1.$

This is due to the fact that the functional $J^{(2)}$ in Formula (4.1.8) does not depend on the choice of $u^{(1)}$.

Similarly, the same function $u^{(2)} = u_0^{(2)}$ delivers the extreme values of the functionals J and $J^{(2)}$ with restrictions:

a) the rate of production growth $\mu\dot{\tilde{z}} = (E_{n-k} - A_4)\tilde{z} - u^{(2)}$;

b) on the initial and final output $\tilde{z}(t_0) = \tilde{z}^0, \qquad \tilde{z}(t_1) = \tilde{z}^1.$

Thus, the original optimization problem is divided into two problems that have smaller dimension systems and fewer restrictions, which are considered in the optimization process that shows the effectiveness of the proposed algorithm with the position of its use in practice.

4.2 Solution of Singularly Perturbed Problem on Optimal Economic Growth

To get one of the possible solutions to this problem

$$\frac{1}{2}b\int_{t_0}^{t_1} e^{-2\delta(t-t_0)}\left(c(t) - \frac{a}{b}\right)^2 dt = \int_{t_0}^{t_1} u^2(t)dt = \|u\|^2_{L_2[t_0,t_1]}, \tag{4.2.1}$$

$$\mu\dot{x} = -\lambda_1 x + F(x) - u, \tag{4.2.2}$$

$$x(t_0) = \sqrt{\frac{b}{2}}k_0 = x_0, \tag{4.2.3}$$

$$x(t_1) = \sqrt{\frac{b}{2}}e^{-\delta(t_1-t_0)}k_1 = x_1, \tag{4.2.4}$$

where the function $f(k)$ has property $f(\alpha k)=\alpha f(k)$ in [85] assume that consumption of worker's and productive reserves is not changed in time. Let us take this assumption for this problem.

In addition, we believe that the rate of discount δ constant and positive, and its value is considered to be quite large, which indicates a greater preference for the useful life of loved ones [82, 85].

If put $\dot{x}=0,$ then from (4.2.2) we get:

$$F\left(x\right)-\lambda_1 x-u=0 \tag{4.2.5}$$

or

$$f\left(k\right)-\lambda_1 k-c=0\,. \tag{4.2.6}$$

If the value k during the time of transition retains its constant value, the control parameter c does not change respectively to k .

By then, the condition $\dfrac{dc}{dk}=0$ follows that

$$f'\left(k\right)=\lambda_1. \tag{4.2.7}$$

Under the made assumptions respectively to the production function $f\left(k\right)$ from (4.2.6), (4.2.7) can only determine the value $k=k^*,\quad c=c^*=f\left(k^*\right)-\lambda_1 k^*$, which satisfy the following inequality:

$$0<c^*<f\left(k^*\right). \tag{4.2.8}$$

Balance at $k\left(t\right)=k^*$, $c\left(t\right)=c^*$ meets all the necessary conditions, except for the boundary conditions

$$k(t_0) = k_0, \tag{4.2.9}$$

$$k(t_1) = k_1. \tag{4.2.10}$$

These values are determined balanced growth mode [85, 93]. With $\mu \to 0$ values k^*, c^* have limits, i.e. $\lim_{\mu \to 0} k^* = \bar{k}^*$, $\lim_{\mu \to 0} c^* = \bar{c}^*$, where $\bar{k}^*, \bar{c}^*$ – solutions of the equations of the form $f'(k) = \varepsilon$, $f(k) - \varepsilon k - c = 0$. On the other hand, the condition $\dot{x}(t) = 0$ we have:

$$\dot{k} - \delta k = 0. \tag{4.2.11}$$

To associate a non-zero solution of (4.2.11) with the point of rest $k = k^*$ we fix one value $t_* \in [t_0, t_1]$. As described above, for any $t_* \in [t_0, t_1]$ is the equality:

$$k(t_*) = k^*. \tag{4.2.12}$$

The solution of the task (4.2.11), (4.2.12) has the form:

$$k(t, k^*) = e^{\delta(t - t^*)} k^*. \tag{4.2.13}$$

This decision at $t \to t^*$ tends to the point of rest k^*. In view of (4.2.13) from the relation $x(t) = \sqrt{\frac{b}{2}} e^{-\delta(t - t_0)} k(t)$,

$$F(x) = f\left(\sqrt{\frac{b}{2}} e^{-\delta(t-t_0)} k(t)\right) - \frac{a}{\sqrt{2b}} e^{-\delta(t-t_0)} = f(x) - \frac{a}{\sqrt{2b}} e^{-\delta(t-t_0)} \tag{4.2.14}$$

we define $x = x^*$, $u = u^*$:

$$x^* = \sqrt{\frac{b}{2}} \cdot e^{-\delta(t^* - t_0)} \cdot k^*, \quad u^* = \sqrt{\frac{b}{2}} e^{-\delta(t^* - t_0)} \left(c^* - \frac{a}{b}\right). \tag{4.2.15}$$

These values satisfy the equation (4.2.5). This point $\tilde{x}^* = \dfrac{F(x^*) - u^*}{\varepsilon}$ is a rest point “connected systems” [20] (in this case one equation)

$$\frac{d\tilde{x}}{d\tau} = -\varepsilon_1 \tilde{x} + F(x^*) - u^*, \quad \tau \geq 0, \tag{4.2.16}$$

$\tilde{x}(0) = x_0$, (or when $\tau \leq 0$)

$$\tilde{x}(0) = x_1. \tag{4.2.17}$$

The point of rest $\tilde{x}^*$ is asymptotically stable in the Lyapunov $\tau \to \infty$. Now the equation (4.2.2) can be rewritten as:

$$x(t,\mu) = e^{-\lambda_1 \frac{t-t_0}{\mu}} x_0 + \frac{1}{\mu}\int_{t_0}^{t} e^{-\lambda_1 \frac{t-s}{\mu}} F(x(s))ds - \frac{1}{\mu}\int_{t_0}^{t} e^{-\lambda_1 \frac{t-s}{\mu}} u(s)ds, \tag{4.2.18}$$

where $\lambda_1 = \lambda_1(\mu) = \varepsilon + \mu(n+\delta)$.

If put in (4.2.2) $F(x(t)) \approx F(x^*)$, then in the known $u = u(t)$ phase coordinates $x(t) = x(t,\mu)$ taking into account the boundary conditions is approximately determined by the formula:

$$x(t,\mu) \approx e^{-\lambda_1\left(\frac{t-t_0}{\mu}\right)} x_0 + \frac{F(x^*)}{\lambda_1}\left(1 - e^{-\lambda_1\left(\frac{t-t_0}{\mu}\right)}\right) - \frac{1}{\mu}\int_{t_0}^{t} e^{-\lambda_1\left(\frac{t-s}{\mu}\right)} u(s)ds. \tag{4.2.19}$$

We agree that the boundary points $x(t_0) = \sqrt{\frac{b}{2}} k_0 = x_0, \quad x(t_1) = \sqrt{\frac{b}{2}} e^{-\delta(t_1 - t_0)} k_1 = x_1$ belong to the domain of influence point of the rest $\tilde{x}^*$. This means that the solution of equation (4.2.16) with the initial condition $\tilde{x}(0) = x_0$ exists for $\tau \geq 0$ and tends to the point of rest $\tilde{x}^*$ at $\tau \to +\infty$, and another solution (4.2.16) with the initial $\tilde{x}(0)=x_1$ also exists for $\tau \le 0$ and tends to the point of rest $\tilde{x}^*$ at $\tau \to -\infty$.

Note. With $\tau \le 0$ from (4.2.17) we obtain the equation with reverse time $\frac{d\tilde{x}}{d\tau}=\varepsilon\tilde{x}-F\left(x^*\right)+u^*, \qquad \tau \le 0.$

In this case, we are interested in is the solution of the problem (4.2.1) (4.2.2) - (4.2.4) which, when $\mu \to 0$ passes to the solution of the problem generator. By (4.2.5), (4.2.15), (4.2.19) of the difference $x(t,\mu)-x^*$ define the following formula:

$$x(t,\mu) \approx e^{-\lambda_1\left(\frac{t-t_0}{\mu}\right)}(x_0-x^*)+x^*-\frac{1}{\mu}\int_{t_0}^{t} e^{-\lambda_1\left(\frac{t-s}{\mu}\right)}u(s)ds. \tag{4.2.20}$$

We define control as follows:

$$u^0(t)=u^0(t,\mu)=V\left(\frac{t-t_0}{\mu}\right), \qquad 0 \le \frac{t-t_0}{\mu} \le \frac{t_1-t_0}{\mu} < +\infty\,. \tag{4.2.21}$$

At $t=t_1$ из (4.2.7) we obtain the following relationship of moment:

$$\alpha_1=\int_0^{\tau_1} e^{-\lambda_1(\tau_1-\tau)}V(\lambda)d\lambda, \tag{4.2.22}$$

where $\alpha_1=-x_1+e^{-\lambda_1\tau_1}x_0+x^*\left(1-e^{-\lambda_1\tau_1}\right)$, $\tau_1=\dfrac{t_1-t_0}{\mu}$.

Then the function $u^0(t)=V(\tau)$ satisfying the relationship of moment (4.2.22), with a minimum rate is:

$$V(\tau)=\frac{2\lambda_1\alpha_1 e^{-\lambda_1(\tau_1-\tau)}}{1-e^{-2\lambda_1\tau_1}}\,. \tag{4.2.23}$$

It should be noted that when $\tau_1-\tau\to\infty$ ($\mu\to 0$) from (4.2.23) we get the following limit relation:

$$\lim_{\substack{\tau_1-\tau\to\infty\\(\mu\to 0)}} V(\tau)=0. \qquad (4.2.24)$$

At the same time, this control $u^0(t,\mu)$ (4.2.21) is optimal in the sense of the task and the corresponding phase coordinates can be written as:

$$x^0(t,\mu)=\frac{F(x^*)-u^*}{\lambda_1}+e^{-\lambda_1\tau}\left(x_0-\frac{F(x^*)-u^*}{\lambda_1}\right)-\frac{\alpha_1 e^{-\lambda_1(\tau_1-\tau)}\left(1-e^{-2\lambda_1\tau}\right)}{1-e^{-2\lambda_1\tau_1}}, \quad 0\le\tau\le\tau_1$$

or

$$x^0(t,\mu)=e^{-\lambda_1\tau}x_0+\tilde{x}^*\left(1-e^{-\lambda_1\tau}\right)-\frac{e^{-\lambda_1(\tau_1-\tau)}\left(1-e^{-2\lambda_1\tau}\right)}{1-e^{-2\lambda_1\tau_1}}\left(-x_1+e^{-\lambda_1\tau_1}x_0+x^*\left(1-e^{-\lambda_1\tau_1}\right)\right), \quad . \qquad (4.2.25)$$

$0\le\tau\le\tau_1$

Function $x(t,\mu)$ (4.2.25) satisfies all the boundary conditions (4.2.3) (4.2.4) at $\mu\to 0\quad(\tau\to\infty)$

$$x^0(t,\mu)\to\frac{F(x^*)-u^*}{\varepsilon}=\tilde{x}^*. \qquad (4.2.26)$$

This means that the point of rest x^* asymptotically stable in the Lyapunov $\tau\to\infty$. The limit relation (4.2.26) can be written in the form:

$$\lim_{\mu\to 0}\sqrt{\frac{b}{2}}e^{-\delta(t-t_0)}k_\mu^0(t,\mu)=\sqrt{\frac{b}{2}}e^{-\delta(t^*-t_0)}\cdot\frac{f(k^*)-c^*}{\varepsilon}, \qquad (4.2.27)$$

where

$$k_\mu^0(t,\mu) = e^{-\delta(t^*-t)}\frac{f(k^*)-c^*}{\lambda_1} + e^{\delta(t-t_0)}\left[e^{-\lambda_1\tau}\left(k_0 - e^{-\delta(t^*-t_0)}\frac{f(k^*)-c^*}{\lambda_1}\right) - \frac{\alpha_1 e^{-\lambda_1(\tau_1-\tau)}\left(1-e^{-2\lambda_1\tau}\right)}{\left(1-e^{-2\lambda_1\tau_1}\right)}\right]. \tag{4.2.28}$$

Function $k^0(t,\mu)$ satisfies all the boundary conditions (4.2.9), (4.2.10), and for its satisfies the following limit relation:

$$\lim_{\substack{\mu\to 0\\ t\to t^*}} k_\mu^0(t,\mu) = \frac{f(k^*)-c^*}{\varepsilon}.$$

In this case, "highway" generates no equilibrium line, and generates a function of the form:

$$\overline{k}_0^*(t,0) = e^{-\delta(t^*-t)}\cdot\frac{f(k^*)-c^*}{\varepsilon}, \qquad t^*\in[t_0,t_1]. \tag{4.2.29}$$

The optimal trajectory, leaving the start point is sent to the "highway" and for quite some time are close to the line (for sufficiently small μ) and goes with it to achieve the desired end state (see. Fig. 4.2.2).

We construct the optimal trajectories $x(t,\mu)$ (4.2.25), $k_\mu^0(t,\mu)$ (4.2.28) и corresponding line (Fig. 4.2.1, 4.2.2) with the following data [85]: the time $t\in[0;21]$; the growth rate of the labor force $n = 0{,}0053$; depreciation rate $\varepsilon = \frac{1}{13} \approx 0{,}0769$; discount coefficient $\delta = 0{,}10$; a small positive parameter $\mu = 0{,}5$ $\lambda = \varepsilon + n = 0{,}0822$, $\lambda_1 = \varepsilon + \mu(n+s) = 0{,}1295$.

The calculations used the following data:

factor of production costs $a = 0,6$, $a_0 = 2,189$, incremental coefficient fond intensity $b = \frac{a_0}{1-a} = 5,4725$, the elasticity of manufacture production assets $\alpha = 0,249$, the elasticity of labor issue $\beta = 0,751$. The initial and final state of the capital-set values for $t = 0$ and $t = 21$: $k(0) = k_0 = 1,8682$, $k(21) = k_1 = 3,7772$, time to highway $t^* = 3$, corresponding capital armament $k^* = 2,0431$.

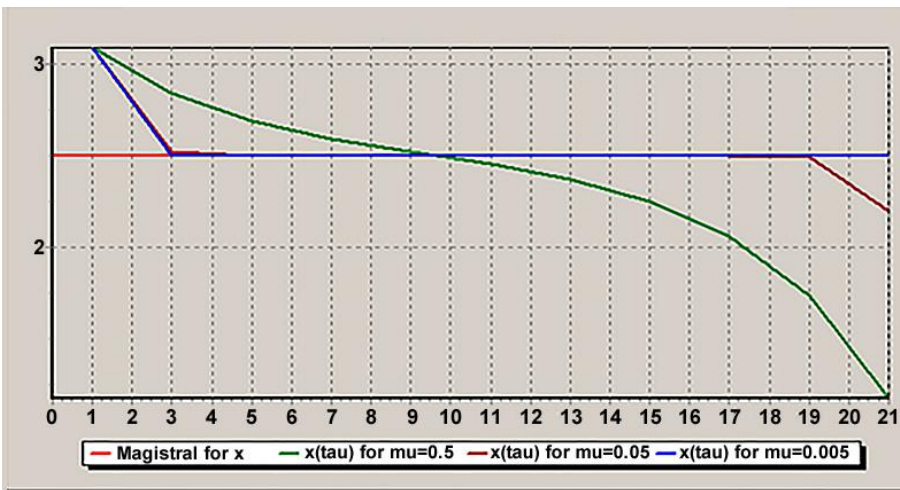


Fig. 4.2.1 *The optimum trajectory for capital intensity of worker* $x(t,\mu)$.

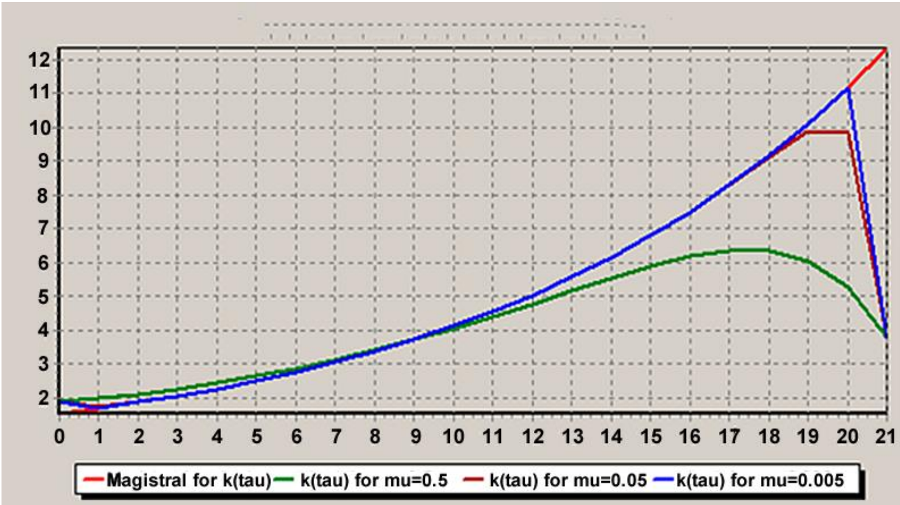


Fig. 4.2.2 *The optimum trajectory for capital intensity of worker* $k(t,\mu)$.

4.3 Control in Single-Commodity Macroeconomic Dynamic Model for Different Optimality Criteria

In here solve two problems of optimal distribution of gross product which illustrates the results obtained in [51, 52].

Problem 1. To solve the problem (dynamic one-commodity model of Leontiev)

$$b\frac{dx}{dt} = (1-a)x - \omega, \qquad x \geq 0\,, \tag{4.3.1}$$

$$x(0) = x_0, \tag{4.3.2}$$

where $x-$ the amount of gross output produced per unit of time; $a-$ coefficient of production material costs; $b-$ coefficient of incremental capital intensity ratio of the time, $\omega-$ consumption. We use the materials [52] i.e. solution of the problem reduces to the solution of the nonlinear Riccati equation (or rather to the solution of Bernoulli's equation) with a final condition.

If we use the terminology introduced in [51, 52] for the formulation of the problem of control, then the relation (4.3.1)

$$\dot{x} = \frac{1-a}{b}x - \frac{1}{b}\omega. \tag{4.3.3}$$

$$0 \le \omega \le (1-a)x \tag{4.3.4}$$

is the equation of state, $\omega -$ is control, $x -$ is state of the system. Now we need to define a control ω, at restrictions (4.3.2) (4.3.3) (4.3.4). The solution to this problem has to satisfy the condition (4.3.4). Existence of such a decision in this case depends primarily on the choice of discount rate δ and the parameter b. Let's show it. Let us consider the function of Hamilton:

$$H = e^{-\delta t}\left\{ p\left(\frac{1-a}{b}x - \frac{1}{b}\omega\right) + \frac{\alpha}{2}\omega^2 \right\}. \tag{4.3.5}$$

Write a necessary condition for an extremum [93]:

$$\frac{\partial H}{\partial \omega} = e^{-\delta t}\left(-\frac{p}{b} + \alpha\omega\right) = 0. \tag{4.3.6}$$

Hence we have:

$$\omega = \frac{p}{\alpha b}. \tag{4.3.7}$$

The canonical equation for the adjoint variable is written as follows:

$$\frac{d}{dt}\left(e^{-\delta t}p(t)\right) = -\frac{\partial H}{\partial x}. \tag{4.3.8}$$

It follows that

$$\dot{p} = -\left(\frac{1-a}{b} - \delta\right)p. \tag{4.3.9}$$

In view of (4.3.7) rewrite (4.3.3):

$$\dot{x} = \frac{1-a}{b}x - \frac{1}{b^2\alpha}p. \tag{4.3.10}$$

We will seek $p(t)$ as:

$$p(t) = K(t)x(t)\,. \tag{4.3.11}$$

Then the expression (4.3.7) is written in the form:

$$\omega = \frac{1}{\alpha b}Kx\,, \tag{4.3.7a}$$

at the time the equation (4.3.10) takes the following form:

$$\dot{x} = \left(\frac{1-a}{b} - \frac{1}{b^2\alpha}K\right)x\,. \tag{4.3.12}$$

From the condition of transversality [93, 95] boundary condition $p(t)$ is given by:

$$p(T) = \beta x(T). \tag{4.3.13}$$

From (4.3.11) it follows that

$$p(T) = K(T)x(T). \tag{4.3.14}$$

Comparing the ratio (4.3.8), (4.3.14), we find that

$$K(T) = \beta. \tag{4.3.15}$$

From the equations (4.3.9) and (4.3.12) it is possible to find the differential equation which must be satisfied by the function $K(t)$. Substituting the equation (4.3.9) the expression (4.3.11) and (4.3.12), we obtain:

$$\dot{K} = -\frac{2(1-a)-\delta b}{b} \quad K + \frac{1}{b^2\alpha}K^2 . \tag{4.3.16}$$

Equation (4.3.16) is a Bernoulli's equation. This equation with the boundary condition (4.3.15) determines uniquely the function $K(t)$. Believing $K(t) \neq 0$ and considering the ratio $\frac{1}{K} = V$, $-\frac{1}{K^2}\dot{K} = \left(\frac{1}{K}\right)' = \dot{V}$ from (4.3.16) we get:

$$\dot{V} = \frac{2(1-a)-\delta b}{b}V - \frac{1}{b^2\alpha}, \quad V(T) = \frac{1}{\beta} . \tag{4.3.17}$$

It is a linear equation and solve it, we obtain the solution of Bernoulli's equation:

$$K(t) = \frac{\beta b\alpha(2(1-a)-b\delta)}{e^{(2(1-a)-b\delta)\left(\frac{t-\mathrm{T}}{b}\right)}\left(b\alpha\left(2(1-a)-b\delta\right)-\beta\right)+\beta} . \tag{4.3.18}$$

It should be noted that the solution of the linear equation (4.3.17) is stable, if the following condition:

$$\frac{2(1-a)}{b} - \delta > 0 \text{ at } t < T$$

or

$$\delta < \frac{2(1-a)}{b} . \tag{4.3.19}$$

In view of (4.3.18), (4.3.11) from (4.3.7) we have:

$$\omega(t) = \frac{\beta(2(1-a)-b\delta)x(t)}{e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\left(b\alpha\left(2(1-a)-b\delta\right)-\beta\right)+\beta} . \tag{4.3.20}$$

Graphic of function $\varpi(t)$ shown in fig. 4.3.2. We obtain the desired solution, and it have to satisfy the limit (1.1.24). It is easy to notice that in this case the parameter b can not be considered to be small, as it tends to zero condition (4.3.4) is not satisfied. From comparison of (4.3.20) and (4.3.4) we have the inequality:

$$\frac{\beta(2(1-a)-b\delta)}{e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}(b\alpha(2(1-a)-b\delta)-\beta)+\beta} \leq 1-a. \tag{4.3.21}$$

If the parameter b you can not tend to zero then we will need to $T \to \infty$. Then (4.3.21), and $T \to \infty$ we get:

$$\frac{1-a}{b} \leq \delta\,. \tag{4.3.22}$$

Combining (4.3.19) and (4.3.22) we have an interval change values of parameter δ:

$$\frac{1-a}{b} \leq \delta < \frac{2(1-a)}{b} \tag{4.3.23}$$

or

$$1 \leq \frac{\delta b}{1-a} < 2\,. \tag{4.3.24}$$

at $t = T$ we have another condition:

$$\frac{\beta}{b\alpha} \leq 1-a\,. \tag{4.3.25}$$

We show that in the interval $0 \leq t \leq T$ function $K(t)$ or left side of (4.3.21) is positive definite. The numerator of the fraction is positive. It follows from (4.3.23). It remains to verify the denominator. Let us assume that the denominator - is positive, i.e.

$$e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\left(b\alpha(2(1-a)-b\delta)-\beta\right)+\beta>0, \qquad (4.3.26)$$

$$\beta\left(1-e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\right)+e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\left(b\alpha(2(1-a)-b\delta)\right)>0.$$

The second term is positive. For the first term was non-negative, it should be

$$1-e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\geq 0$$

or

$$e^{(2(1-a)-b\delta)\left(\frac{t-T}{b}\right)}\leq 1. \qquad (4.3.27)$$

This inequality holds for all $t\leq T$.

The phase variable $x(t)$, taking into account (4.3.20), is given by:

$$x(t)=e^{\frac{1-a}{b}t}\cdot\frac{(1-a)\left(2-\frac{b\delta}{1-a}\right)+\frac{\beta}{b\alpha}\left(e^{\frac{1-a}{b}\left(2-\frac{b\delta}{1-a}\right)(T-t)}-1\right)}{(1-a)\left(2-\frac{b\delta}{1-a}\right)+\frac{\beta}{b\alpha}\left(e^{\frac{1-a}{b}\left(2-\frac{b\delta}{1-a}\right)T}-1\right)}x_0 . \qquad (4.3.28)$$

On the fig. 4.3.1 is a charting function.

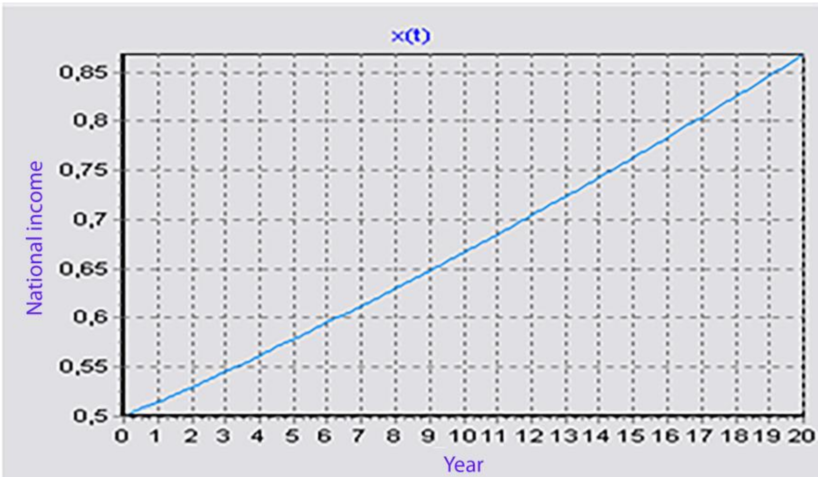


Fig. 4.3.1 *Schedule of function* $x(t)$ *(formula 4.3.28).*

So, we have indicated the conditions (4.3.27), under which there is a solution to this problem in the form (4.3.28). On the fig. 4.3.3 is shown the optimal regulator of system.

"Amplification coefficient" $K(t)$ obtained by modeling equation (4.3.17).

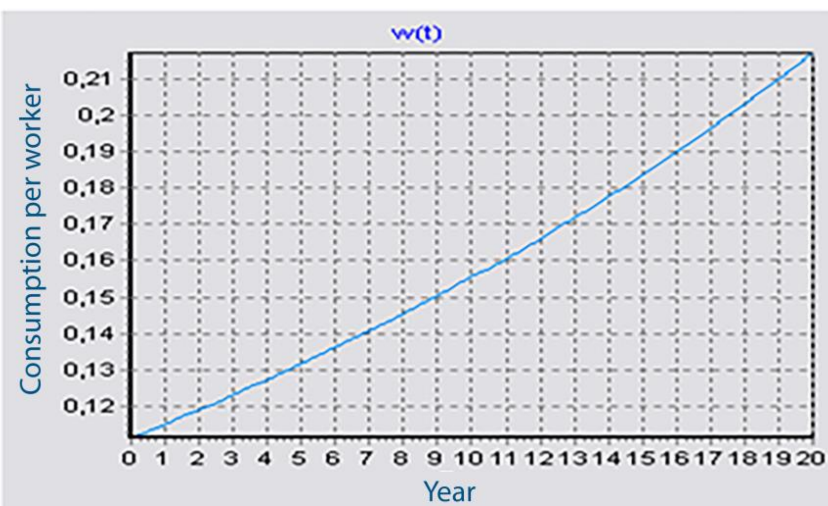


Fig. 4.3.2 *Schedule of function* $\omega(t)$ *(formula 4.3.20).*

Problem 2**.** Now consider another task. In this case the quality of the process is estimated linear functional:

$$J = \int_0^T e^{-\delta t} w(t)dt \to \max \qquad (4.3.29)$$

or

$$J = -\int_0^T e^{-\delta t} w(t)dt \to \min\ . \qquad (4.3.29)$$

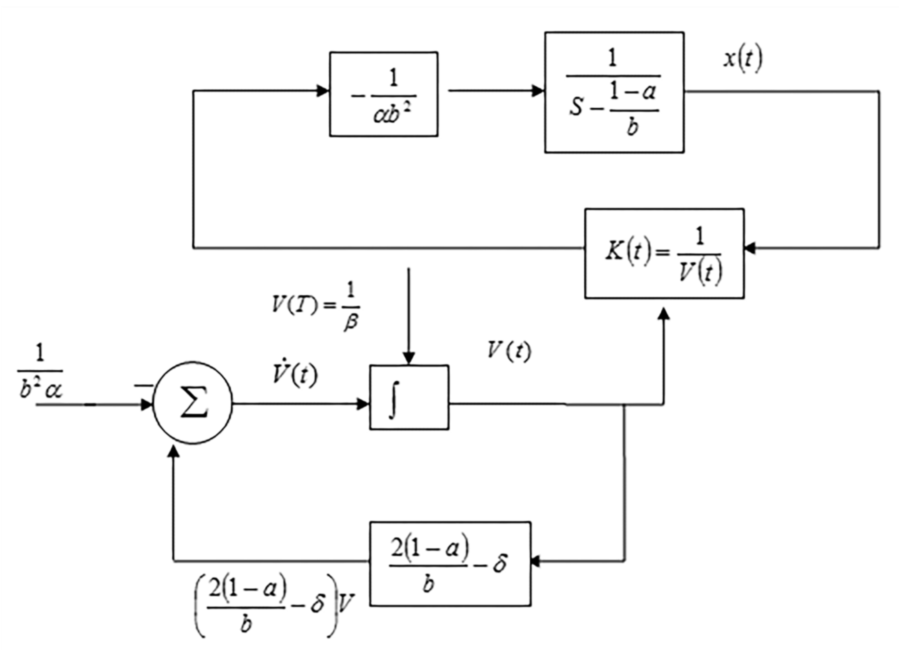


Fig. 4.3.3 *Optimum regulator of system* $\dot{x}(t) = \frac{1-a}{b}x(t) - \frac{1}{b}\omega(t)$.

We also assume the intensity of consumption $w(t)$ can not exceed a certain maximum level w^*, so that

$$0 \le w \le w^*\ . \qquad (4.3.30)$$

For solve this problem we use the method set out in [41]. Considering the consequent economic sense variables (they are not negative), we can not take the type of restriction

$$\max_{t_0 \le t \le T} \left| u(t) \right| \le \nu . \tag{4.3.31}$$

Therefore, we need to introduce a new function to get artificially limiting the type (4.3.31) from (4.3.30). So, enter the following function:

$$u = w - \frac{w^*}{2} . \tag{4.3.32}$$

Then in view of (4.3.32) from (4.3.30), we obtain:

$$-\frac{w^*}{2} \le u \le \frac{w^*}{2} \tag{4.3.33}$$

or

$$\left| u(t) \right| \le \frac{w^*}{2} . \tag{4.3.34}$$

Now we need to change the dynamic, respectively, one-commodity model Leontiev, which represents the balance ratio [86]

$$b\frac{dx}{dt} = (1-a)x - \omega, \qquad x \ge 0 , \tag{4.3.35}$$

$$x(0) = x_0 , \tag{4.3.36}$$

where $x -$ the amount of gross output produced per unit of time; $a -$ coefficient of production material costs; $b -$ coefficient of incremental capital intensity ratio of the time, $\omega -$ consumption:

$$b\frac{dx}{dt} = (1-a)x - u - \frac{w^*}{2}, \quad x \ge 0, \tag{4.3.37}$$

$$x(0) = x_0 \,. \tag{4.3.38}$$

Horizon process time distribution of gross product is considered as final. Then it is necessary to set the final time the minimum acceptable value of the intensity of the gross output in order to allow consumption and outside this time horizon:

$$x(T) = x_T \,. \tag{4.3.39}$$

Then we need to solve the following problem: find a control $u(t)$, minimizes the function (4.3.29) with restrictions (4.3.34) - (4.3.39).

This problem reduces to the problem of moments. Solution of the equation (4.3.37) with the initial condition (4.3.38) can be written in the form:

$$x(t) = e^{\left(\frac{1-a}{b}\right)t} x_0 - \frac{1}{b}\int_0^T e^{\left(\frac{1-a}{b}\right)(t-s)} u(s)\,ds + \frac{w^*}{2(1-a)}\left(1 - e^{\left(\frac{1-a}{b}\right)t}\right). \tag{4.3.40}$$

Hence, when at $t = T$ we will have:

$$\alpha = \int_0^T e^{-\left(\frac{1-a}{b}\right)t} u(t)\,dt, \tag{4.3.41}$$

where $\alpha = b\left(-x_T + e^{\left(\frac{1-a}{b}\right)T} x_0 + \frac{w^*}{2(1-a)}\left(1 - e^{\left(\frac{1-a}{b}\right)T}\right)\right)$.

Therefore, we need to find a control $u(t)$, which minimizes the functional (4.3.29) with restrictions (4.3.34), (4.3.41).

According to [51] ($u^0(t) = \nu sign\left\{ B_0'(t)\Phi'(T,t) l_1^{(0)} + B_2'(T) e^{-A_4'(t)\left(\frac{T-t}{\mu}\right)} l_2^{(0)} - e^{-\delta t} \right\}$),

the control is written in the form:

$$u(t)=\frac{w^*}{2}sign\left(\lambda_1 e^{-\left(\frac{1-a}{b}\right)(t-T)}-e^{-\delta t}\right).,\qquad (4.3.42)$$

function $\lambda_1 e^{-\left(\frac{1-a}{b}\right)(t-T)}-e^{-\delta t}$ can change sign more than once.

Therefore, setting $\lambda_1 e^{-\left(\frac{1-a}{b}\right)(s-T)}-e^{-\delta s}=0,$ we find:

$$s=\frac{1}{\frac{1-a}{b}-\delta}\left(\ln\lambda_1+\frac{1-a}{b}T\right).\qquad (4.3.43)$$

Believing $s\in(0,T)$ and substituting (3.6.42) into (4.3.41), we obtain the following relations:

$$s=\frac{1}{\frac{1-a}{b}-\delta}\left(\ln\lambda_1+\frac{1-a}{b}T\right),\qquad (4.3.44)$$

$$\lambda_1=e^{-\frac{1-a}{b}T}\left\{\frac{1}{2}\left(1+e^{-\left(\frac{1-a}{b}\right)T}\right)-\frac{\alpha(1-a)}{w^*b}e^{-\left(\frac{1-a}{b}\right)T}\right\}^{\frac{b\delta}{1-a}-1}.\qquad (4.3.45)$$

According to [51], control (4.3.42) have to satisfy the following moment ratio

$$\int_0^T e^{-\delta t}u(t)\,dt=\alpha\lambda_1-1.\qquad (4.3.46)$$

Substituting (4.3.45) to (4.3.43), we obtain:

$$s=-T-\frac{b}{1-a-b\delta}\ln\left(\frac{1}{2}(e^{(\frac{1-a}{b})T}+1)-\frac{\alpha(1-a)}{w^*b}\right),\qquad (4.3.47)$$

in order to $s\in(0,T)$ have to have the following conditions:

$$0 < \frac{b\delta}{1-a} < 1, \tag{4.3.48}$$

$$\frac{1}{2}\left(e^{(\frac{1-a}{b}T)}+1\right)-\frac{\alpha(1-a)}{w^*b} > 1. \tag{4.3.49}$$

From (4.3.49), we have:

$$|\alpha| < \frac{w^*b}{2(1-a)}\left(e^{\frac{1-a}{b}T}-1\right). \tag{4.3.50}$$

So we have a necessary condition for the existence of control $u(t)$, that satisfies the relations of moment (4.3.41), (4.3.46) and limitation (4.3.34).

Now substituting control (4.3.42) to the equation (4.3.46) and taking into account the relations (4.3.44), (4.3.45), we obtain the following equation for the parameter λ_1:

$$2e^{\frac{\delta T}{1-\frac{b\delta}{1-\alpha}}}(1-\frac{b\delta}{1-a})-(\frac{2\delta}{w^*}+e^{-\delta T}-1)\lambda_1^{\frac{\frac{b\delta}{1-a}}{1-\frac{b\delta}{1-a}}}+\frac{b\delta}{1-a}(e^{(\frac{1-a}{b})T}+1)\lambda_1^{\frac{1}{1-\frac{b\delta}{1-a}}}=0. \tag{4.3.51}$$

We introduce the notation $\mu = \frac{b\delta}{1-a}$. Then the equation (4.3.51) can be written as:

$$\mu\left(e^{\frac{1-a}{b}T}+1\right)\lambda_1^{\frac{1}{1-\mu}}-\left(\frac{2\delta}{w^*}+e^{-\delta T}+1\right)\lambda_1^{\frac{\mu}{1-\mu}}+2e^{\frac{\delta T}{1-\mu}}(1-\mu)=0. \tag{4.3.52}$$

For different values μ satisfying (4.3.48) we have the equation of a different nature. So at $\mu \neq 1$ and natural $n = \frac{\mu}{1-\mu}$ available equation of at least second order. Believing $n = \frac{\mu}{1-\mu} = 1$, we find $\mu = \frac{1}{2}$. This value satisfies the condition

(4.3.48). Then, to determine the parameters λ_1 we obtain the following quadratic equation

$$\left(e^{\left(\frac{1-a}{b}\right)T}+1\right)\lambda_1^2-2\left(\frac{2\delta}{w^*}+e^{-\delta T}+1\right)\lambda_1+2e^{2\delta T}=0. \qquad (4.3.53)$$

This equation has real solutions, if the condition

$$\left(\frac{2\delta}{w^*}+e^{-\delta T}+1\right)^2-2e^{2\delta T}\left(e^{\left(\frac{1-a}{b}\right)T}+1\right)\geq 0. \qquad (4.3.54)$$

For to select $b,\quad \delta$ we have the following relationship:

$$\delta=\frac{1-a}{2b} \text{ or}$$

$$b=\frac{1-a}{2\delta}. \qquad (4.3.55)$$

The positive solution of the equation (4.3.53) is the desired parameter optimal control (4.3.42). After determining the desired value $\lambda_1=\lambda_1^*$, solution of the this problem can be written as:

$$w=\frac{w^*}{2}+\frac{w^*}{2}sign\left(\lambda_1^*e^{-\left(\frac{1-a}{b}\right)(t-T)}-e^{-\delta t}\right). \qquad (4.3.56)$$

Consumption w defined in the form (3.6.56) satisfies the restriction (4.3.30). Then under consideration on the interval will have the following path:

$$
x(t)=\begin{cases}
e^{(\frac{1-a}{b})t}\left(x_0-\dfrac{w^*}{1-a}\right)+\dfrac{w^*}{1-a}, & 0\le t<s,\\[2ex]
\dfrac{\dfrac{w^*}{1-a}x_T}{x_T-e^{(\frac{1-a}{b})T}\left(x_0-\dfrac{w^*}{1-a}\right)}, & t=s,\\[2ex]
e^{(\frac{1-a}{b})(t-T)}\cdot x_T, & s<t\le T,
\end{cases}
$$

where $s=T-\dfrac{2b}{1-a}\ln\dfrac{1-a}{w^*}\left(\cancel{x_T}-e^{(\frac{1-a}{b})T}x_0-\dfrac{w^*}{1-a}\right).$

The graph of this function and the table are shown in fig. 4.3.4.

It should be noted that in this case the process of distribution of gross domestic product is characterized by a singularly perturbed equation, since the (4.3.35) can be written as

$$\mu\frac{dx}{dt}=\delta x-\frac{\delta\omega}{1-a}, \tag{4.3.57}$$

where $\mu=\dfrac{b\delta}{1-a}<1-$ small parameter.

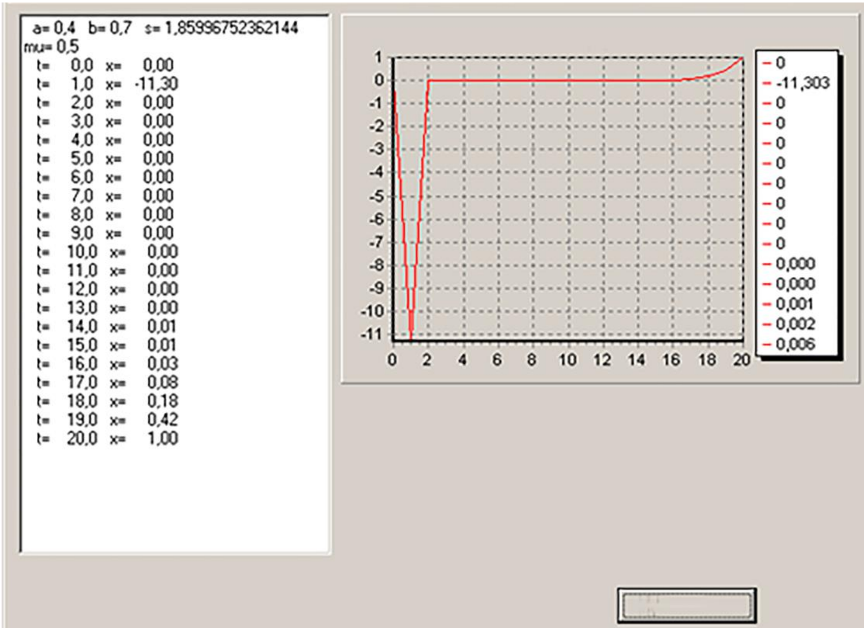


Fig. 4.3.4 *Graph of function* $x(t)$ *and table of values.*

4.4 Investigation of the Problem on Optimal Control in Single-Commodity Macro Models Based on the Delay of the Process of Investments

As decomposition described in chapter 1, for the task

$$\dot{k} = -(\varepsilon + n)k + \upsilon, \tag{4.4.1}$$

$$\mu\dot{\upsilon} = (1-a)\, fk - (1+\mu n)\upsilon - \bar{w},$$

where $\mu = \dfrac{1}{\lambda}$, $\quad \bar{w} = w - (1-a)r$ we will share the slow and fast position.

Matrix A and B have the form:

$$A = \begin{pmatrix} -(\varepsilon+n) & 1 \\ \dfrac{(1-a)\,f}{\mu} & -\dfrac{1+\mu n}{\mu} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

In this case

$$A_1 = -(\varepsilon + n), \qquad A_2 = 1, \qquad A_3 = (1-a) f,$$
$$A_4 = -(1+\mu n), \qquad B_1 = 0, \qquad B_2 = -1.$$

Then the system with separated variables can be written as:

$$\begin{aligned} \dot{\tilde{k}}_\mu &= m_1 \tilde{k}_\mu - N \bar{w}_\mu, \\ \mu \dot{\tilde{\upsilon}}_\mu &= m_2 \tilde{\upsilon}_\mu - \bar{w}_\mu, \end{aligned} \tag{4.4.2}$$

where

$$m_1 = -(\varepsilon + n) + \frac{\varepsilon\mu - 1 + \sqrt{(\mu\varepsilon - 1)^2 + 4(1-a) f \mu}}{2\mu},$$
$$m_2 = -(1+\mu n) - \frac{\varepsilon\mu - 1 + \sqrt{(\mu\varepsilon - 1)^2 + 4(1-a) f \mu}}{2}$$

$$N = \frac{1}{\sqrt{(\mu\varepsilon - 1)^2 + 4(1-a) f \mu}}, \quad \tilde{\upsilon}_\mu = \upsilon_\mu - H_1 k_\mu, \quad \tilde{k}_\mu = k_\mu + \mu N \tilde{\upsilon}_\mu,$$

$$\bar{w}_\mu = w_\mu - (1-a) r, \quad w_\mu(t) = w(t,\mu), \quad \tilde{\upsilon}_\mu(t) = \tilde{\upsilon}(t,\mu), \quad \tilde{k}_\mu(t) = \tilde{k}(t,\mu).$$

For the system (4.4.2), we have the following boundary conditions:

$$\tilde{k}_\mu(0) = \tilde{k}_0, \qquad \tilde{\upsilon}_\mu(0) = \tilde{\upsilon}_0, \tag{4.4.3}$$

$$\tilde{k}_\mu(T) = \tilde{k}_T, \qquad \tilde{\upsilon}_\mu(T) = \tilde{\upsilon}_T, \tag{4.4.4}$$

where
$$\begin{aligned} \tilde{k}_0 &= k_0 + \mu N \tilde{\upsilon}_0, \qquad & \tilde{\upsilon}_0 &= \upsilon_0 - H_1 k_0, \\ \tilde{k}_T &= k_T + \mu N \tilde{\upsilon}_T, \qquad & \tilde{\upsilon}_T &= \upsilon_T - H_1 k_T, \end{aligned}$$

$$H_1 = \frac{\varepsilon\mu - 1 + \sqrt{(\mu\varepsilon - 1)^2 + 4(1-a) f \mu}}{2\mu}, \quad \lim_{\mu \to 0} H_1(\mu) = (1-a) f.$$

It is easy can to make sure that inequality

$$\left(1-a\right) f < \left(\varepsilon + n\right)\left(1 + \mu n\right) \tag{4.4.5}$$

is a condition of the sustainability systems (4.4.2).

For the system (4.4.1) with the boundary conditions (4.4.3), (4.4.4) consider the problem of choosing a trajectory of consumption per worker at the condition

$$J = -\int_0^T e^{-\delta s} w_{\varpi}\left(s\right) ds \to \min, \tag{4.4.6}$$

or $J = \int_0^T e^{-\delta s} w_{\varpi}\left(s\right) ds \to \max.$

In the future, we believe that for the system (4.4.2) run the condition (4.4.5), i.e. the system is stable. In this problem, the phase coordinates are the basic production assets for the one worker k and volume of investments was put into effect, calculated per worker υ and consumption per worker w - is a control parameter. It should be noted that the consumption values can not less than zero and in the closed economy can not rise higher output per worker, i.e.

$$0 \le w \le fk_* + r, \tag{4.4.7}$$

where $k_* -$ the maximum allowable level of capital per worker.

Solution of the system (4.4.2) with the initial conditions (4.4.3) can be written as:

$$\tilde{k}_\mu(t) = e^{m_1 t}\tilde{k}_0 - \int_0^t e^{m_1(t-s)} N\bar{w}_\mu(s)ds,$$

$$\tilde{\upsilon}_\mu(t) = e^{m_2 \frac{t}{\mu}}\tilde{\upsilon}_0 - \frac{1}{\mu}\int_0^t e^{m_2 \frac{(t-s)}{\mu}} \bar{w}_\mu(s)ds. \tag{4.4.8}$$

At $t=T$, taking into account the final conditions (4.4.4) from (4.4.8) we have the following relation of moment:

$$c_1=\int_0^T e^{m_1(T-s)}N\bar{w}_\mu(s)ds,\ \ c_2=\frac{1}{\mu}\int_0^T e^{m_2(\frac{T-s}{\mu})}\bar{w}_\mu(s)ds, \tag{4.4.9}$$

where $c_1=-\tilde{k}_T+e^{m_1T}\tilde{k}_0,\quad c_2=-\tilde{\upsilon}_T+e^{m_2\frac{T}{\mu}}\tilde{\upsilon}_0.$

Considering that $\bar{w}_\mu=w_\mu-(1-a)r$ and from (4.4.6) we have:

$$J=\int_0^T e^{-\delta s}\bar{w}_\mu(s)ds+\int_0^T e^{-\delta s}(1-a)rds\ . \tag{4.4.10}$$

The second integral does not depend on $\bar{w}_\mu$ and the desired minimum of $\bar{w}_\mu$ value $J(\bar{w}_\mu)$ will be reached on the same functions $\bar{w}_\mu=\bar{w}_{\mu_{onm}}(t)$, and that the minimum of expression

$$J_1=\int_0^T e^{-\delta s}\bar{w}_\mu(s)ds. \tag{4.4.11}$$

From (4.4.7) we have the following restrictions for function $\bar{w}_\mu$:

$$0\le\bar{w}_\mu\le fk_*+r. \tag{4.4.12}$$

Then

$$0\le\int_0^T\bar{w}_\mu^2(t)dt\le l, \tag{4.4.13}$$

where $l=(fk_*+r)^2T.$

Thus, have a problem about the minimum of (4.4.11) with restrictions (4.4.9), (4.4.13).

A similar task can be solved by the method of A. I. Egorov [37], which is based on theorem of Levi about orthogonal decomposition of elements of a Hilbert space. The procedure for constructing an optimal control method of A. I. Egorov differs little from of the same scheme as in the method of moments [53]. But it is convenient for the practical construction of optimal control, it is not required to solve the auxiliary extreme problems. Desired control take in the form:

$$\bar{w}_\mu\left(t\right)=\gamma_1 e^{m_1(T-t)}N+\gamma_2 e^{m_2\left(\frac{T-t}{\mu}\right)}+\gamma_0 e^{-\delta t}. \qquad (4.4.14)$$

This control belongs to a boundary of the set (4.4.13) [96]:

$$\int_0^T \bar{w}_\mu^2\left(t\right)dt=l. \qquad (4.4.15)$$

Constants $\gamma_1,\gamma_2,\gamma_0$ will be determined from the relation (4.4.9), (4.4.15). Substituting the value of $\bar{w}(t)$ from formula (4.4.14) to the relation (4.4.9), (4.4.15), we obtain a system of algebraic equations

$$r_{11}\gamma_1+r_{12}\gamma_2+r_{13}\gamma_0=c_1,\ r_{21}\gamma_1+r_{22}\gamma_2+r_{23}\gamma_0=c_2, \qquad (4.4.16)$$

$$r_{11}\gamma_1^2+r_{22}\gamma_2^2+r_{33}\gamma_0^2+2r_{12}\gamma_1\gamma_2+2r_{13}\gamma_1\gamma_0+2r_{23}\gamma_2\gamma_0=l,$$

where $r_{11}=\dfrac{N^2}{2m_1}(e^{2m_1T}-1),\quad r_{12}=\dfrac{N\mu}{\mu m_1+m_2}(e^{(m_1\mu+m_2)\frac{T}{\mu}}-1),\quad r_{13}=\dfrac{e^{m_1T}N}{m_1+\delta}(1-e^{-(m_1+\delta)T}),$

$$\mu r_{21}=r_{12},\ r_{22}=\frac{1}{2m_2}(e^{\frac{2m_2T}{\mu}}-1),\ r_{23}=\frac{e^{\frac{m_2T}{\mu}}}{m_2+\mu\delta}(1-e^{-\frac{m_2+\delta\mu}{\mu}T}),\ r_{33}=\frac{1}{2\delta}(1-e^{-2\delta T}).$$

Of the first two equations of the systems (4.4.15) γ_1,γ_2 are uniquely determined through γ_0:

$$\gamma_1 = d_1\gamma_0 + h_1,\ \gamma_2 = d_2\gamma_0 + h_2, \tag{4.4.17}$$

where $d_1 = \dfrac{r_{12}r_{23} - r_{13}r_{22}}{r_{11}r_{22} - r_{21}r_{12}}$, $h_1 = \dfrac{c_1 r_{22} - c_2 r_{12}}{r_{11}r_{22} - r_{12}r_{21}}$, $d_2 = \dfrac{r_{12}r_{13} - r_{11}r_{23}}{r_{11}r_{22} - r_{12}r_{21}}$,

$h_2 = \dfrac{c_2 r_{11} - c_1 r_{12}}{r_{11}r_{22} - r_{12}r_{21}}$, $r_{11}r_{22} - r_{12}r_{21} \neq 0.$

Substituting the value of γ_1 and the last equation of the system (4.4.16), get the quadratic equation with respect γ_0:

$$a\gamma_0^2 + 2b\gamma_0 + c = 0, \tag{4.4.18}$$

where $a = r_{11}d_1^2 + r_{22}d_2^2 + r_{33} + 2r_{12}d_1d_2 + 2r_{13}d_1 + 2r_{23}d_2$,

$$b = r_{11}d_1h_1 + r_{22}d_2h_2 + r_{12}d_1h_2 + r_{12}d_2h_1 + r_{13}h_1 + r_{23}h_2,$$
$$c = r_{11}h_1^2 + r_{22}h_2^2 + 2r_{12}h_1h_2 - l.$$

Equation (4.4.18) has two real roots $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$, if they performed condition

$$b^2 - ac \geq 0. \tag{4.4.19}$$

Substituting the found values γ_0 in (4.4.17), obtain:

$$\gamma_1^{(1)} = d_1\gamma_0^{(1)} + h_1,\ \gamma_1^{(2)} = d_1\gamma_0^{(2)} + h_1,\ \gamma_2^{(1)} = d_2\gamma_0^{(1)} + h_2,$$
$$\gamma_2^{(2)} = d_2\gamma_0^{(2)} + h_2. \tag{4.4.20}$$

Then by the expression (4.4.14) we define two functions

$$\overline{w}_\mu^{(1)}(t) = \gamma_1^{(1)}e^{m_1(T-t)}N + \gamma_2^{(1)}e^{m_2(\frac{T-t}{\mu})} + \gamma_0^{(1)}e^{-\delta t},$$

$$\overline{w}_\mu^{(2)}(t) = \gamma_1^{(2)}e^{m_1(T-t)}N + \gamma_2^{(2)}e^{m_2(\frac{T-t}{\mu})} + \gamma_0^{(2)}e^{-\delta t}. \tag{4.4.21}$$

One of them is the solution of the considered task. Everything, from these controls by substituting in functional (4.4.11), we will define the optimum decision which corresponds to the minimum value of functional. We write down optimal control as

$$\overline{w}_{\mu_{onm}}(t) = \gamma_1^{(\mu)} e^{m_1(T-t)} N + \gamma_2^{(\mu)} e^{m_2 \frac{(T-t)}{\mu}} + \gamma_0^{(\mu)} e^{-\delta t}, \qquad (4.4.22)$$

where $\gamma_0^{(\mu)}, \gamma_1^{(\mu)}, \gamma_2^{(\mu)} -$ optimum coefficients which depend on small parameter. Then the phase coordinates corresponding to optimum control (4.4.22) are defined by ratios:

$$k_{\mu_{onm}}(t) = e^{m_1 t}\tilde{k}_0 - \mu N \tilde{\upsilon}_0 + \frac{N^2 e^{m_1 T}}{2m_1}\left(e^{-m_1 t} - e^{m_1 t}\right)\gamma_1^{(\mu)} + \frac{e^{\frac{m_2 T}{\mu}} N\mu}{\mu m_1 + m_2}\left(e^{-\frac{m_2 t}{\mu}} - e^{m_1 t}\right)\gamma_2^{(\mu)} + \\ + \frac{N}{m_1 + \delta}\left(e^{-\delta t} - e^{m_1 t}\right)\gamma_0^{(\mu)}, \qquad (4.4.23)$$

$$\upsilon_{\mu_{onm}}(t) = e^{\frac{m_2 t}{\mu}}\tilde{\upsilon}_0 - Hk_{\mu_{onm}}(t) + \frac{Ne^{m_1 T}}{m_1\mu + m_2}(e^{-m_1 t} - e^{\frac{m_2 t}{\mu}})\gamma_1^{(\mu)} - \frac{e^{\frac{m_2(T+t)}{\mu}}}{2m_2}(1 - e^{-\frac{2m_2 t}{\mu}})\gamma_2^{(\mu)} + \\ \frac{1}{\mu\delta + m_2}(e^{-\delta t} - e^{\frac{m_2 t}{\mu}})\gamma_0^{(\mu)}. \qquad (4.4.24)$$

On fig. 4.4.1 are graphs of control $\overline{w}_{\mu_{onm}}(t)$ and functions $\tilde{k}_{\mu_{onm}}(t)$, $\tilde{\upsilon}_{\mu_{onm}}(t)$ respectively at $\mu = 0{,}5$. At $\mu \to 0$ we have the following limit ratios:

$$m_1 \to -(\varepsilon + n) + (1-a)f = m_0, \;\; m_2 \to -1,$$

$$N \to 1, \; H_1 \to (1-a)f, \; r_{11} \to r_{11}^{(0)} = \frac{1}{2m_0}(e^{2m_0 T} - 1),$$

$$r_{12} \to r_{12}^{(0)} = 0, \quad r_{21} \to r_{21}^{(0)} = 0,$$

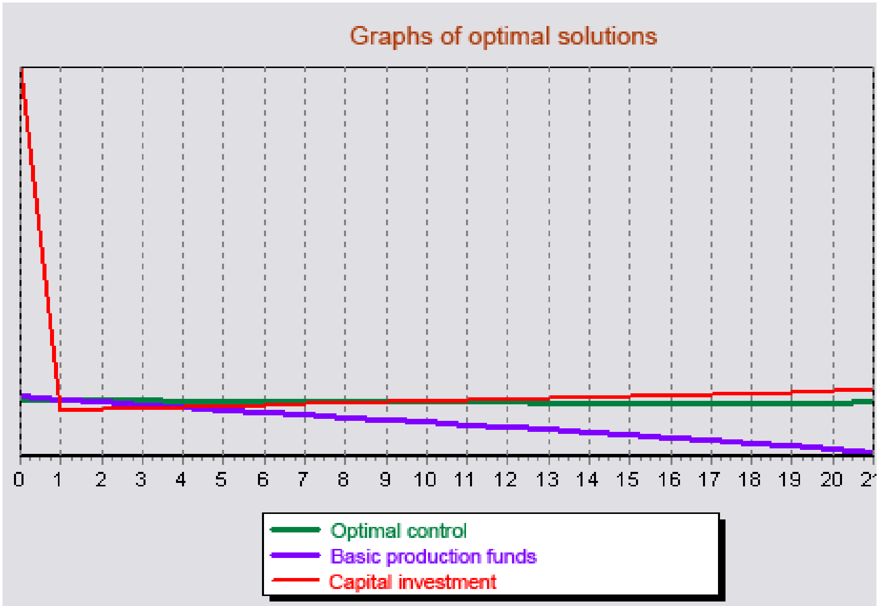


Fig. 4.4.1 *Graph of optimum decisions.*

$$r_{13} \to r_{13}^{(0)} = \frac{e^{m_0 T}}{m_0 + \delta}(1 - e^{-(m_0+\delta)T}),\, r_{22} \to r_{22}^{(0)} = \frac{1}{2},\, r_{23} \to r_{23}^{(0)} = e^{-\delta T},$$

$$r_{33} = r_{33}^{(0)} = \frac{1}{2\delta}(1 - e^{-2\delta T}),$$

$$d_1 \to d_1^{(0)} = -\frac{r_{13}^{(0)}}{r_{11}^{(0)}},\, d_2 \to d_2^{(0)} = -2r_{23}^{(0)},\; c_1 \to c_1^{(0)} = -k_T + e^{m_0 T} k_0, \quad (4.4.25)$$

$$c_2 \to c_2^{(0)} = -\upsilon_T + (1-a)\, f k_T,\; h_1 \to h_1^{(0)} = \frac{c_1^{(0)}}{r_{11}^{(0)}},\; h_2 \to h_2^{(0)} = 2c_2^{(0)},\; r_{11}^{(0)} \neq 0.$$

For limit values of parameters from (4.4.25) it is possible to receive the ratios similar (4.4.18) - (4.4.21) and to define from them limit values of optimum coefficients: $\gamma_0^{(\mu)} \to \gamma_0^{(0)}$, $\gamma_1^{(\mu)} \to \gamma_1^{(0)}$, $\gamma_2^{(\mu)} \to \gamma_2^{(0)}$. Then from (4.4.22), (4.4.23) and (4.4.24) at $\mu \to 0$, because of the system (4.4.2) stability we receive the following limit ratios of the solution of a task respectively (see fig. 4.4.2)

$$w^{(0)}(t) = e^{m_0(T-t)}\gamma_1^{(0)} + e^{-\delta t}\gamma_0^{(0)} + (1-a)r\,, \qquad (4.4.26)$$

$$k^{(0)}(t) = e^{m_0 t}k_0 + \frac{e^{m_0 T}}{2m_0}(e^{-m_0 t} - e^{m_0 t})\gamma_1^{(0)} + \frac{1}{m_0+\delta}(e^{-\delta t} - e^{m_0 t})\gamma_0^{(0)}, \quad (4.4.27)$$

$$\upsilon^{(0)}(t) = k^{(0)}(t)\cdot(1-a)f - \overline{w}^{(0)}(t) = (1-a)(fk^{(0)}(t) + r) - w^{(0)}(t). \quad (4.4.28)$$

We will notice that direct application the method's of A. I. Egorov [37] to this task increases number of the equations by one. If the studied system has high dimension, the method will be ineffective. Therefore we pass on another.

We will consider system

$$\dot{k}_\mu = m_0 k_\mu - \overline{w}_\mu^*,\ \ k_\mu(0) = k_0\,,\ k_\mu(T) = k_T\,, \qquad (4.4.29)$$

$$\mu\dot{\upsilon}_\mu^* = -\upsilon_\mu^* - \overline{w}_\mu^*,\ \ \upsilon_\mu^*(0) = \upsilon_0 - (1-a)f\kappa_0 = \upsilon_0^*,\ \ \upsilon_\mu^*(T) = \upsilon_\mathrm{T} - (1-a)fk_T = \upsilon_T^*,$$

where $\upsilon_\mu^* = \upsilon_\mu - (1-a)fk_\mu$.

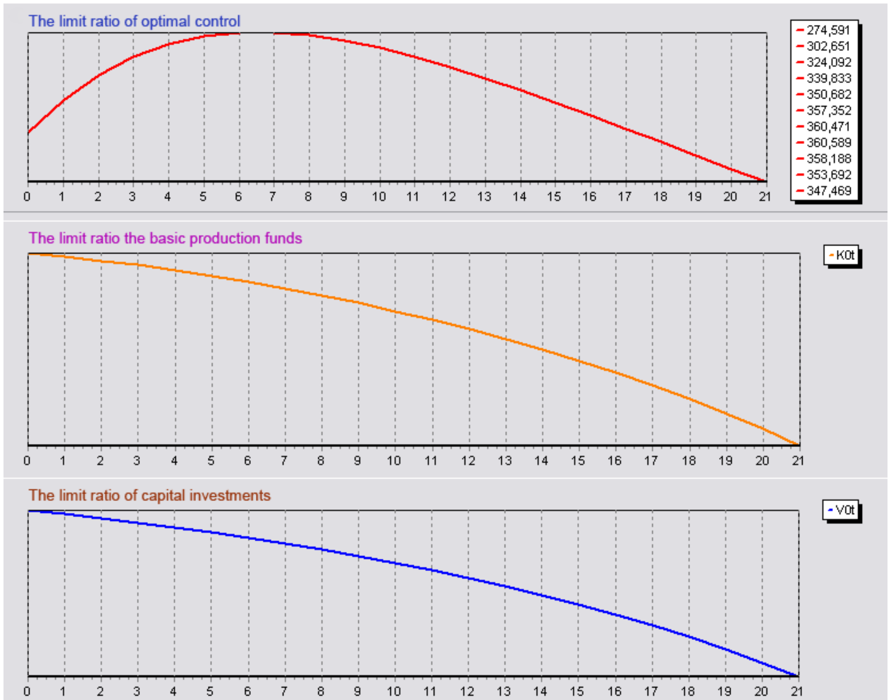


Fig. 4.4 *Limit ratios.*

Now we will consider a task about a minimum of functional (4.4.11) at restrictions (4.4.29) and (4.4.13). We will designate this task a symbol P_{μ}. At $\mu = 0$ from (4.4.29) we receive the generating system

$$\dot{k}^{(0)} = m_0 k^{(0)} - \overline{w}^{(0)}, \quad k^0(0) = k_0, \quad k^{(0)}(T) = k_T, \tag{4.4.30}$$

$$0 = -\upsilon^* - \overline{w}^{(0)} \text{ or } \upsilon^{(0)} = (1-a)\, fk^{(0)} - \overline{w}^{(0)},$$

where $\upsilon^* = \upsilon^{(0)} - (1-a)\, fk^{(0)}$.

Then tasks (4.4.11), (4.4.13), (4.4.30) are limit in relation to a task P_{μ}. We will designate it through P_0.

We solve the task P_0. We write down the decision of system (4.4.30) in a look:

$$k^0(t) = e^{m_0 t} k_0 - \int_0^t e^{m_0(t-s)} \overline{w}^{(0)}(s)ds, \tag{4.4.31}$$

$$\upsilon^{(0)}(t) = (1-a) f k^{(0)}(t) - \overline{w}^{(0)}(t).$$

At $t = T$ from the first equation (4.4.31), we obtain:

$$c_1^{(0)} = \int_0^T e^{m_0(T-s)} \overline{w}^{(0)}(s)ds, \tag{4.4.32}$$

where $c_1^{(0)} = -k_T + e^{m_0 T} k_0$.

Similarly, above stated method, control $\overline{w}^{(0)}(t)$ will be search in the form of:

$$\overline{w}^{(0)}(t) = \overline{\gamma}_1 e^{m_o(T-t)} + \overline{\gamma}_0 e^{-\delta t}. \tag{4.4.33}$$

This control belongs to the boundary of the set (4.4.13), i.e. it must satisfy the equation

$$\int_0^T \overline{w}^{(0)^2}(t)dt = l. \tag{4.4.34}$$

Given the (4.6.33) by (4.4.32), (4.6.34), we obtain the following system of algebraic equations $\overline{\gamma}_1, \overline{\gamma}_0$:

$$\begin{cases} r_{11}^{(0)} \overline{\gamma}_1 + r_{13}^{(0)} \overline{\gamma}_0 = c_1^{(0)}, \\ r_{11}^{(0)} \overline{\gamma}_1^2 + 2 r_{13}^{(0)} \overline{\gamma}_1 \overline{\gamma}_0 + r_{33}^{(0)} \overline{\gamma}_0^2 = l, \end{cases} \tag{4.4.35}$$

where $r_{11}^{(0)}$, $r_{13}^{(0)}$, $r_{33}^{(0)}$ defined by the relations of formula (4.4.25). From the first equation (4.6.35), we obtain:

$$\bar{\gamma}_1 = \bar{d}_1 \bar{\gamma}_0 + \bar{h}_1, \tag{4.4.36}$$

where $\bar{d}_1 = -\frac{r_{13}^{(0)}}{r_{11}^{(0)}}$, $\bar{h}_1 = \frac{c_1^{(0)}}{r_{11}^{(0)}}$. Given the (4.6.36), from the second equation (4.6.35), we obtain a quadratic equation for $\bar{\gamma}_0$:

$$\bar{a}\bar{\gamma}_0^2 + 2\bar{b}\,\bar{\gamma}_0 + \bar{c} = 0, \tag{4.4.37}$$

where $\bar{a} = r_{11}^{(0)}\bar{d}_1^2 + 2r_{13}^{(0)}\bar{d}_1 + r_{33}^{(0)}$, $\bar{b} = r_{11}^{(0)}\bar{d}_1\bar{h}_1 + r_{13}^{(0)}\bar{h}_1$, $\bar{c} = r_{11}^{(0)}\bar{h}_1^2 - l.$

Equation (4.6.37) has two real roots $\bar{\gamma}_0^{(1)}$, $\bar{\gamma}_0^2$, if $\bar{b}^2 - \bar{a}\bar{c} \geq 0$. Assume that these roots exist. From (4.6.33) we have two functions -control, one of which minimizes the functional (4.4.11), written in the form (see. fig.4.4.3)

$$w^{(0)}(t) = \bar{\gamma}_1^{(0)} e^{m_0(T-t)} + \bar{\gamma}_0^{(0)} e^{-\delta t} + (1-a)r\,, \tag{4.4.38}$$

where $\bar{\gamma}_1^{(0)}$, $\bar{\gamma}_0^{(0)}$ – optimal coefficients. Substituting (4.4.38) to (4.4.31), we obtain (fig. 4.4.3):

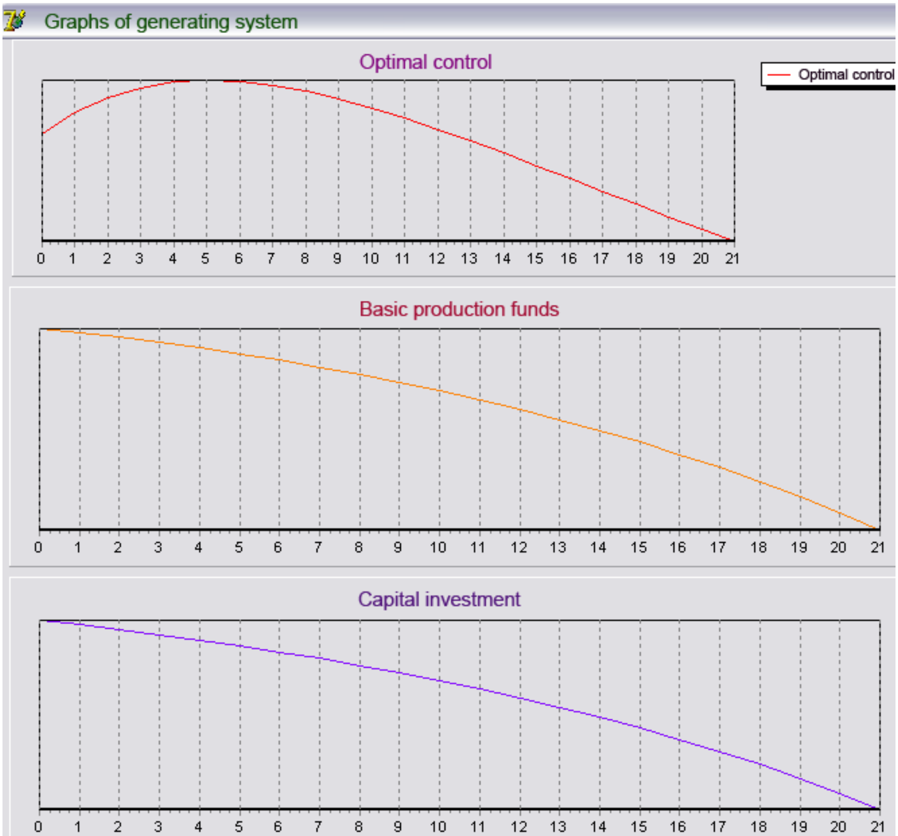


Fig. 4.4.3 *The graphs generating systems.*

$$k^{(0)}(t) = e^{m_0 t} k_0 + \frac{e^{m_0 T}}{2m_0}(e^{-m_0 t} - e^{m_0 t})\bar{\gamma}_1^{(0)} + \frac{1}{m_0 + \delta}(e^{-\delta t} - e^{m_0 t})\bar{\gamma}_0^{(0)}, \qquad (4.4.39)$$

$$\upsilon^{(0)}(t) = -\overline{w}^{(0)}(t) + (1-a) f \cdot k^{(0)}(t) = (1-a)(fk^{(0)}(t) + r) - w^{(0)}(t).$$

So, we got a solution to the problem P_0.

Note that the resulting function in (4.6.38), (4.6.39) coincide with the functions in formulas (4.6.26) - (4.4.28) (see. fig.4.4.2).

We arrive at the solution of the tasks P_μ. We define control $\overline{w}_\mu^*(t)$ in shape:

$$\overline{w}_{\mu}^{*}(t)=\begin{cases}\overline{w}^{(0)}(t), & 0\le t\le T,\\ \eta\left(\dfrac{T-t}{\mu}\right), & 0\le\dfrac{T-t}{\mu}\le\dfrac{T}{\mu}<\infty,\end{cases}\tag{4.4.40}$$

where $\eta\left(\dfrac{T-t}{\mu}\right)$ – the function of boundary layer type, which have is exponential decrease. The function $\overline{w}_0(t)$ already identified, remains to be determined $\eta\left(\dfrac{T-t}{\mu}\right)$. With $\overline{w}_{\mu}^{*}(t)=\eta\left(\dfrac{T-t}{\mu}\right)$ from the second equation (4.4.29) we have:

$$\upsilon_{\mu}^{*}(t)=e^{-\frac{t}{\mu}}\left(\upsilon_0^{*}+\upsilon^{(0)}(0)\right)-\upsilon^{(0)}(t)-\frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\eta\left(\frac{T-s}{\mu}\right)ds\,.\tag{4.4.41}$$

With $t=T$, from (4.4.41) we have the following relation of moment:

$$c_2^{*}(t)=\frac{1}{\mu}\int_0^T e^{-\frac{T-s}{\mu}}\eta\left(\frac{T-s}{\mu}\right)ds,\tag{4.4.42}$$

where $c_2^{*}=-\upsilon_T^{*}+e^{-\frac{T}{\mu}}\left(\upsilon_0^{*}+\upsilon^{(0)}(0)\right)-\upsilon^{(0)}(T)$.

Repeating the above procedure in this case, we define control $\overline{w}_{\mu}^{*}(t)=\eta\left(\dfrac{T-t}{\mu}\right)$ so that it is minimize the functional (4.4.11) and satisfy to relation of moment equation (4.6.42). As we saw above, such control exists on the boundary of the set (4.4.13) and getting in shape:

$$\eta\left(\frac{T-t}{\mu}\right)=\beta_1^{*}e^{-\frac{T-t}{\mu}}+\mu\beta_0^{*}e^{-\delta t}.\tag{4.4.43}$$

Optimal coefficients β_1^*, β_0^* selected from solutions of the system algebraic equations:

$$\begin{aligned} &r_{22}^*\beta_1 + \mu r_{23}^*\beta_0 = c_2^*, \\ &r_{22}^*\mu\beta_1^{\ 2} + 2\mu^2 r_{23}^*\beta_1\beta_0 + r_{33}^*\mu^2\beta_0^{\ 2} = l, \end{aligned} \tag{4.4.44}$$

where $r_{22}^* = \lim\limits_{m_2\to -1} r_{22} = \dfrac{1}{2}\left(1 - e^{-\frac{2T}{\mu}}\right)$, $\quad r_{23}^* = \lim\limits_{m_2\to -1} r_{23} = \dfrac{1}{1-\delta\mu}\left(e^{-\delta T} - e^{-\frac{T}{\mu}}\right)$, $\quad r_{33}^* = r_{33}$.

From the first equation (4.6.44), we obtain a quadratic equation with respect to β_0^*:

$$p\beta_0^{*2} - r = 0, \tag{4.4.45}$$

where $p = r_{33}\mu^2 - \mu^3\dfrac{r_{23}^2}{r_{22}}$, $\quad r = l - \mu\dfrac{c_2^{*2}}{r_{22}}$. Equation (4.6.45) has two real roots $\beta_0^{*(1)}$ and $\beta_0^{*(2)}$. Assume that these roots exist. Then from (4.6.43) we have two functions-control, one of which minimizes the functional (4.4.11).

Substituting control (4.4.43) to (4.6.41) we obtain:

$$\upsilon_\mu^* = e^{-\frac{t}{\mu}}\left(\upsilon_0^* + \upsilon^{(0)}(0)\right) - \upsilon^{(0)}(t) - \frac{1}{2}\beta_1^* e^{-\frac{T+t}{\mu}}\left(1 - e^{\frac{2t}{\mu}}\right) - \frac{\beta_0^*\mu}{1-\delta\mu}\left(e^{-\delta t} - e^{-\frac{t}{\mu}}\right).$$

Given the (4.6.38), (4.6.43) control $\overline{w}_\mu^*(t)$ represented in the form:

$$\overline{w}_\mu^*(t) = \begin{cases} \gamma_1^{(0)} e^{m_0(T-t)} + \gamma_0^{(0)} e^{-\delta t}, & 0 \le t \le T, \\ \beta_1^* e^{-\tau} + \mu\beta_0^* e^{-\delta\mu(\tau_1 - \tau)}, & 0 \le \tau \le \tau_1 < +\infty, \end{cases}$$

where $\tau = \dfrac{T-t}{\mu}$, $\quad \tau_1 = \dfrac{T}{\mu}$.

Three $(\overline{w}_{\mu}^{*}, k^{(0)}, \upsilon_{\mu}^{*})$ is a solution the problem. It should be noted that the system (4.4.29) approximates the system (4.4.2) with an accuracy $O(\mu)$. In addition, when $\mu \to 0$ the solution of problem P_{μ} approaches to solution of P_0 , which was obtained in (4.6.39) (4.6.40) (or (4.6.26) - (4.4.28)). Consequently, the solution of problem P_{μ} is an asymptotic approximation of the solution of the initial problem with the accuracy of the order of smallness $O(\mu)$. When the delay at enter capital investments disappears, i.e. at $\mu \to 0$ we have the solution of problem P_0. The solution of the problem P_0 forms an arterial road. The specific investment imposed by into action without delays, described by the function $\upsilon^{0}(t)$. In this case, we arrive to a certain idealization, but in fact the process of development of investments in the economy without delays occurs. However, the study duration of the process assimilation of capital investments in a certain period of time and the effect of the lag (delay) on other indicators of economic growth of interest of a scientific nature. In this sense, the convergence rate for solving the problem to the solution of the problem P_0 is of practical importance.

4.5 Estimation of Optimal Development of the Economy Based on a Single-Commodity Optimization Model of with a Small Parameter

Here, by the method small parameter is investigated the optimal control problem for single-commodity model of economy and conducted a comparative analysis with the known results which receive from [85] in other ways. It should be noted that at the beginning of this task is required forming arterial road of this model and to find corresponding control realizing the arterial road [20] at solving the problem (4.3.29).

$$J_1 = \int_0^T e^{-\delta t}(1-a)uxdt \to \max \qquad (4.3.29)$$

or

$$J_1 = -\int_0^T e^{-\delta t}(1-a)uxdt \to \min\,. \qquad (4.3.29)$$

Then the reduced problem can be formulated as follows: to find such process $\upsilon = (k(t), X(t), u(t))$, which minimizes the (4.3.29) at restrictions

$$\mu\dot{k} = -(\varepsilon + n)k + (1-a)(1-u)x, \qquad (4.5)$$

$$k(0) = k_0\,,\ k(T) = k_1\,,$$

$$0 \le u \le 1,\ \ x = f(k,t) \ge 0\,,$$

$$k(t) \ge k_3(t)\,,\ 0 \le \mu \le 1,$$

where $f(k,t) = \frac{1}{L}F(k,L,t)$.

Here we propose a new approach, which is based on simple properties decreasing function on a closed interval. We denote by D the set of values k, satisfies (4.5) and call it the permissible region the process. A similar problem in the case of $\mu = 1$ considered in [85].

Highest average consumption, which should be secured by this process is estimated by the value of functional (4.3.29) with the sign reversed. In this problem, the state of the system is k - the amount of capital per worker, control - labor productivity x and the share of consumption u . The equation the process is the differential equation of growth capital intensity.

If enter a "rapid" time τ by formula $\tau = \dfrac{t}{\mu}$, where $\mu -$ small parameter, then time τ the original t it is a "slow" time. In such a case, the variables coefficients of the of the studied system in "fast" time τ will be slowly varying. Administering to a system small parameter μ - it is a certain idealization, which emphasizes the fact that the pace course of the process above (approximately, $\dfrac{1}{\mu}$ time increases) than in the normal mode.

Let the size of the final product are determined by the production function of the Cobb-Douglas [82, 85]. Then labor productivity x is determined by the function

$$x = be^{\rho t}k^{\alpha},\ \alpha = 1-\beta, \tag{4.5.1}$$

where ρ - coefficient defining rate of growth of a technical process, α - the coefficient of elasticity of manufacture production assets; β -the coefficient of elasticity of the release of labor.

Equation (4.5) with (4.5.1) is written as:

$$\mu\dot{k} = -(\varepsilon + n)k + (1-a)(1-u)be^{\rho t}k^{\alpha}. \tag{4.5.2}$$

Consider the problem (4.3.29), (4.5.2). We introduce a new function

$$V = ke^{-\delta t}. \tag{4.5.3}$$

Then in view of (4.5.3) from (4.5.2) we have:

$$\mu\frac{dV}{dt} = (-(\varepsilon + n + \delta\mu)k + (1-a)be^{\rho t}k^{\alpha})e^{-\delta t} - e^{-\delta t}b(1-a)e^{\rho t}k^{\alpha}u. \tag{4.5.4}$$

Now, from the right side of (4.5.4) delete k. We require that the sum of standing to the $e^{-\delta t}$, i.e. function

$$m(k,t) = -(\varepsilon + n + \delta\mu)k + (1-a)be^{\rho t}k^{\alpha}$$

did not depend of k. Then

$$\frac{\partial m}{\partial k} = -(\varepsilon + n + \delta\mu) + (1-a)be^{\rho t}\alpha k^{\alpha-1} = 0 .$$

From here we have:

$$k^{\alpha-1} = \frac{\varepsilon + n + \delta\mu}{\alpha(1-a)be^{\rho t}} = k^{-\beta} \tag{4.5.5}$$

or

$$k = k_{\mu}^{*} = \left(\frac{\alpha(1-a)b}{\varepsilon + n + \delta\mu}\right)^{\frac{1}{\beta}} e^{\frac{\rho}{\beta}t} . \tag{4.5.6}$$

Using by (4.5.3) the equation (4.5.4) can be written in the form

$$\mu\frac{dV}{dt} = (-(\varepsilon + n + \delta\mu) + (1-a)be^{\rho t}k^{\alpha-1})V - b(1-a)e^{\rho t}uk^{\alpha-1}V . \tag{4.5.7}$$

Than in view of (4.5.5) from (4.5.7) we will obtain:

$$\mu\frac{dV}{dt} = -\frac{\varepsilon + n + \delta\mu}{\alpha}(u - \beta)V . \tag{4.5.8}$$

In view of (4.5.5) the functional (4.3.29) can be written as:

$$J = -\frac{\varepsilon + n + \delta\mu}{\alpha}\int_0^T uVdt . \tag{4.5.9}$$

Taking into account (4.5.6) from (4.5.3) we have:

$$V = \left(\frac{\alpha(1-a)b}{\varepsilon + n + \delta\mu}\right)^{\frac{1}{\beta}} e^{-(\delta-\frac{\rho}{\beta})t} = e^{-(\delta-\frac{\rho}{\beta})t}V(0), \tag{4.5.10}$$

$$V(0) = k(0) = \left(\frac{\alpha(1-a)b}{\varepsilon + n + \delta\mu} \right)^{\frac{1}{\beta}}.$$

From formulas (4.5.10) we find:

$$\dot{V} = -(\delta - \frac{\rho}{\beta})V. \qquad (4.5.11)$$

Comparing equation (4.5.8), (4.5.11), we obtain:

$$\frac{\varepsilon + n + \delta\mu}{\alpha}(u - \beta) = \mu(\delta - \frac{\rho}{\beta}). \qquad (4.5.12)$$

Condition (4.5.12) takes place if

$$u = u_\mu^* = 1 - \alpha \frac{\frac{\rho}{\beta}\mu + \varepsilon + n}{\varepsilon + n + \delta\mu}. \qquad (4.5.13)$$

Similarly in [85], the function $k_\mu^*(t)$ (4.5.6) call *highway* of the dynamic model. Control, implementing this feedline - a constant, which is determined by (4.5.13). Then in view of (4.5.13) equation (4.5.9) is written as:

$$J = -\frac{\varepsilon + n + \delta\mu}{\alpha} \left(1 - \alpha \frac{\frac{\rho}{\beta}\mu + \varepsilon + n}{\varepsilon + n + \delta\mu} \right) \int_0^T V(t)dt.$$

Here the integrand $V(t) = e^{-\delta t} k_\mu^*(t) = e^{-\left(\delta - \frac{\rho}{\beta}\right)t} \cdot V(0)$ - the discounted value of the capital.

From (4.5.10) can to see that at $\delta > \frac{\rho}{\beta}$ the function $V(t)$ - decreasing on the segment $[0, T]$, and $V(0)$ is its largest value, i.e.

$$T \cdot e^{-\left(\delta-\frac{\rho}{\beta}\right)T} V(0) \le \int_0^T V(t)dt \le T \cdot V(0).$$

Where, $J_\mu^* = J_{\min} = -\dfrac{\varepsilon+n+\delta\mu}{\alpha}\left(1-\alpha\dfrac{\frac{\rho}{\beta}\mu+\varepsilon+n}{\varepsilon+n+\delta\mu}\right)\cdot V(0)\cdot T.$

With $\delta < \dfrac{\rho}{\beta}$ function $V(t)-$ increasing on the segment $[0,T]$. Then

$$T \cdot V(0) \le \int_0^T V(t)dt \le T \cdot e^{\left(\frac{\rho}{\beta}-\delta\right)T} \cdot V(0),$$

$$J_\mu^* = J_{\min} = -\frac{\varepsilon+n+\delta\mu}{\alpha}\left(1-\alpha\frac{\frac{\rho}{\beta}+\varepsilon+n}{\varepsilon+n+\delta\mu}\right)\cdot T \cdot e^{\left(\frac{\rho}{\beta}-\delta\right)\cdot T} \cdot V(0). \qquad (4.5.14)$$

With $\delta = \dfrac{\rho}{\beta}$ we have

$$J_\mu^* = J_{\min} = -\frac{\varepsilon+n+\delta\mu}{\alpha}\beta \cdot T \cdot V(0). \qquad (4.5.15)$$

In the "fast" time τ highway is

$$k_\mu^* = k_\mu^*(\tau\mu) = \left(\frac{\alpha(1-a)b}{\varepsilon+n+\delta\mu}\right)^{\frac{1}{\beta}} e^{\frac{\rho}{\beta}\tau\mu} \qquad (4.5.16)$$

And it will be slowly varying functions. With $\mu \to 0$, k_μ^*, u_μ^*, J_μ^* have the following limit values

$$k_\mu^* \to k_0^* = \left(\frac{\alpha(1-a)b}{\varepsilon+n}\right)^{\frac{1}{\beta}},$$

$$u_\mu^* \to u_0^* = \beta, \quad J_\mu^* \to J_0^* = -\frac{\varepsilon+n}{\alpha}\beta V(0)\cdot T.$$

For short periods of time changing the "slow" variables does not affect the fast equations and consequently limit values are k_0^*, u_0^* may serve as an asymptotic approximation when forming highway and enables to obtain a qualitative picture of it.

In order to process υ was optimal in the sense of the solution of the task k_μ^* it should satisfy the given boundary conditions (4.5). But it not so the solution k_μ^* can not satisfy the boundary conditions (4.5) because through these points are other curves, which are partial solutions of the original equation (4.5.2) with a predetermined control u.

We define these curves and their points of intersection with the highway (switching point) k_μ^*. Dividing both sides of the differential equation (4.5) on k^α we have:

$$\mu k^{-\alpha}\dot{k} = -(\varepsilon+n)k^{1-\alpha} + (1-a)(1-u)be^{\rho t}.$$

We introduce a new variable

$$k^{1-\alpha} = k^\beta = z. \tag{4.5.17}$$

Then, taking into account (4.5.16) from (4.5.15) we get:

$$\mu\dot{z} = -\beta(\varepsilon+n)z + \beta(1-a)(1-u)be^{\rho t}. \tag{4.5.18}$$

With the known u (u -constant) exact solution (4.5.18) with the initial condition $z(0)=k_0^\beta$ written in the form of the Cauchy formula:

$$z(t)=e^{-\beta(\varepsilon+n)\frac{t}{\mu}}k_0^\beta+\frac{\beta}{\mu}(1-a)(1-u)b\int_0^t e^{-\beta(\varepsilon+n)\frac{t-s}{\mu}}e^{\rho s}ds$$

or

$$z(t)=e^{-\beta(\varepsilon+n)\frac{t}{\mu}}\left(k_0^\beta-\frac{a_0(1-u)}{\varepsilon+n+\mu\dfrac{\rho}{\beta}}\right)+\frac{a_0(1-u)}{\varepsilon+n+\mu\dfrac{\rho}{\beta}}e^{\rho t}, \qquad (4.5.19)$$

where $a_0=b(1-a)$.

Similarly, the solution of (4.5.18) with the initial condition $z(T)=k_1^\beta$ is determined by the relationship:

$$z(t)=e^{-\beta(\varepsilon+n)\frac{t-T}{\mu}}\left(k_1^\beta-\frac{a_0(1-u)e^{\rho T}}{\varepsilon+n+\mu\dfrac{\rho}{\beta}}\right)+\frac{a_0(1-u)e^{\rho t}}{\varepsilon+n+\mu\dfrac{\rho}{\beta}}.$$

Note that if for the given problem build the Hamiltonian function, then it will depend on the control u linearly and its maximum value are achieved only in the boundary values. But in the real economic problems, as noted in [85], the minimum level of consumption is strictly positive: $0<u_*\le u\le 1$. Therefore, the Hamiltonian takes the maximum value in points $u=u_*,\quad u=1$ and through these values can be determined switching point.

For to determine the point of intersection of the highway with the boundaries of the permissible region D we have the following relations:

$$\frac{a_0\alpha}{\lambda+\delta\mu}e^{\rho t}=e^{-\beta\lambda\frac{t}{\mu}}\left(k_0^{\beta}-\frac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}\right)+\frac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}e^{\rho t}, \qquad (4.5.20)$$

$$\frac{a_0\alpha}{\lambda+\delta\mu}e^{\rho t}=e^{-\beta\lambda\frac{t-T}{\mu}}\left(k_1^{\beta}-\frac{a_0(1-u_i)e^{\rho T}}{\lambda+\mu\dfrac{\rho}{\beta}}\right)+\frac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}e^{\rho t}, \qquad (4.5.21)$$

where $\lambda=\varepsilon+n,\qquad a_0=b(1-a)$, $i=1{,}2$.

In formulas (4.5.20), (5.6.21) if $i=1,$ it takes the lower limit $u=u_1=u_*$, if $i=2$, then $u=u_2=1$. Then the left and right switching points are calculated by the following formulas:

$$t_1=-\frac{\mu}{\beta\lambda+\mu\rho}\ln\frac{\dfrac{a_0\alpha}{\lambda+\delta\mu}-\dfrac{(1-u_i)a_0}{\lambda+\mu\dfrac{\rho}{\beta}}}{k_0^{\beta}-\dfrac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}},$$

$$t_2=\frac{\beta\lambda\mathrm{T}}{\beta\lambda+\mu\rho}-\frac{\mu}{\beta\lambda+\mu\rho}\ln\frac{\dfrac{a_0\alpha}{\lambda+\delta\mu}-\dfrac{(1-u_i)a_0}{\lambda+\mu\dfrac{\rho}{\beta}}}{k_1^{\beta}-\dfrac{a_0(1-u_i)}{\lambda+\mu\dfrac{\rho}{\beta}}e^{\rho\mathrm{T}}}.$$

The boundaries of the permissible region D is defined by the relations (4.5.17), (4.5.19) at values $u=u_*,\quad u=1$. Let's $k_0<k_\mu^*(0),\quad k_1>k_\mu^*(T)$. Then highway $k_\mu(t)$ (4.5.15) proceeds as shown in fig. 4.5.1. As can be seen from the figure, the optimal trajectory consists of three sections with moments of switching t_1 and t_2. Starting from the time t_1 until t_2, up growth is on highway, and outside the interval (t_1, t_2) consumption is at a lower level u^*, i.e. in those periods of time in the economy is a process of accumulation. The output trajectory to a highway at different values of the small parameter is shown in fig. 4.5.3-4.5.6.

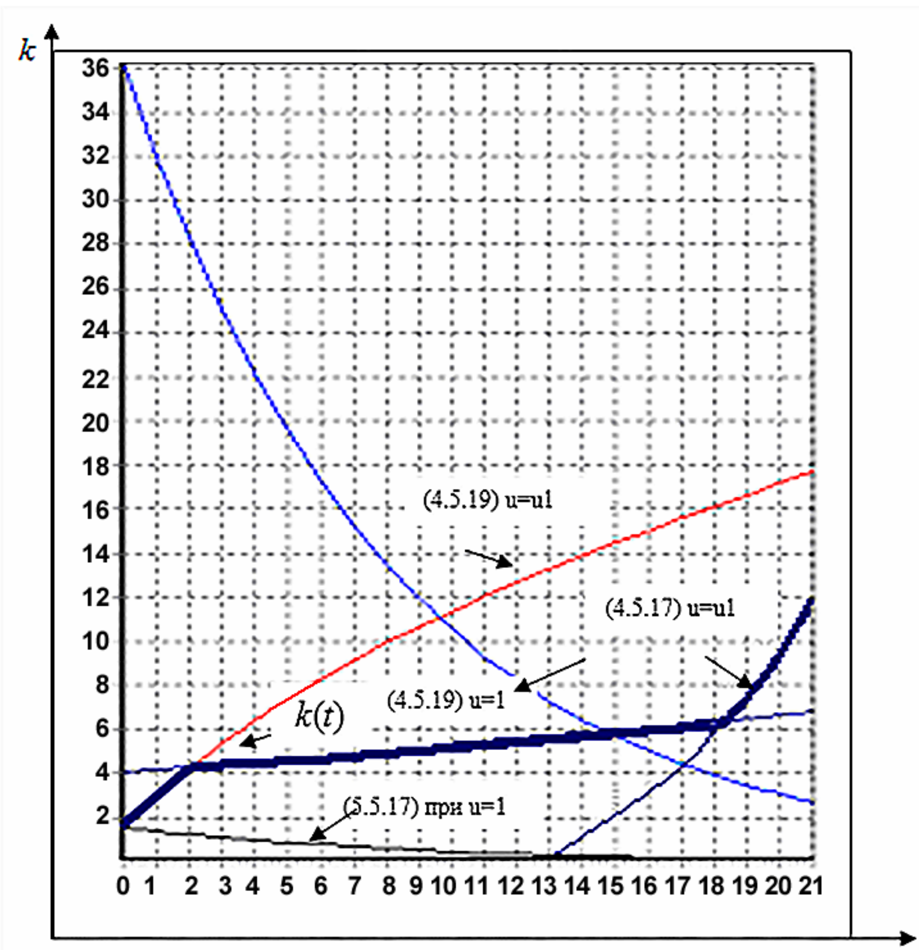


Fig. 4.5.1 *The optimal trajectory the moment of switching.*

As we noted above, that the small parameter is introduced artificially into the system, so that the result was a simplified algorithm that allows us to offer cost-effective computational procedures. Therefore, we need to derive the corresponding asymptotic formulas that make it possible to build the optimal trajectory with a certain precision, while maintaining the qualitative features of the processes under study. Moving on to "rapid" time $\tau = \dfrac{t}{\mu}$ make the change of variable in (4.5.18):

$$\frac{dz}{d\tau} = -\beta\lambda z + \beta a_0 (1-u), \; z(0) = k_0^{\beta}. \tag{4.5.22}$$

The solution of equation (4.5.22) if known u is as follows:

$$z(\tau) = e^{-\beta\lambda\tau}\left(k_0^{\beta} - \frac{a_0(1-u)}{\lambda}\right) + \frac{a_0(1-u)}{\lambda}. \tag{4.5.23}$$

Graph of the function is shown in fig.4.5.7. For $\sigma = \dfrac{t-T}{\mu}$ $(t > T)$ from (4.5.18) will have:

$$\frac{dz}{d\sigma} = -\beta\lambda z + \beta a_0 (1-u)e^{\rho T} \quad z(0) = k_1^{\beta}. \tag{4.5.24}$$

The solution (4.5.24) can be written as:

$$z(\sigma) = e^{-\beta\lambda\sigma}\,(k_1^{\beta} - \frac{a_0(1-u)}{\lambda}e^{\rho T}) + \frac{a_0(1-u)e^{\rho T}}{\lambda}. \tag{4.5.25}$$

A graph of this function is shown in fig. 4.5.2.

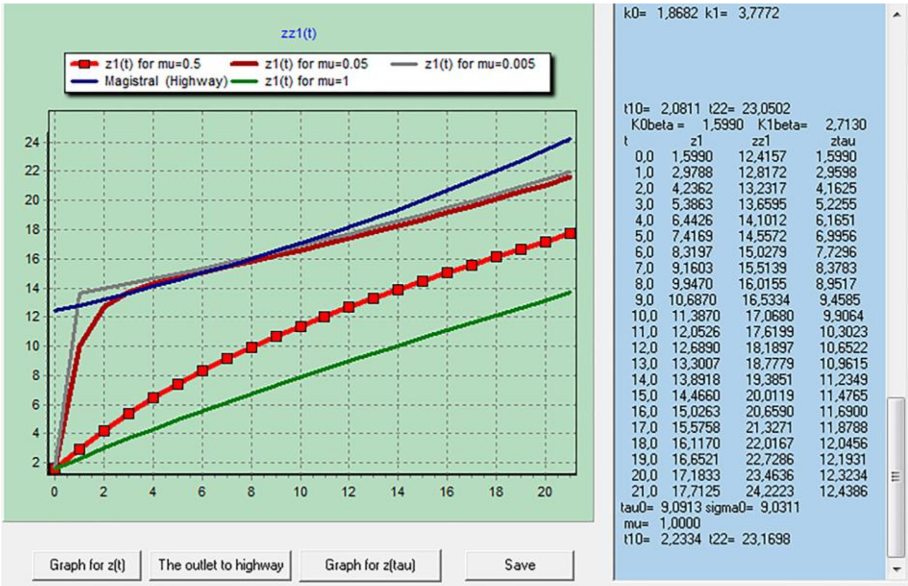


***Fig. 4.5.2** Graphics functions* $z(t)$, $z(\sigma)$ *and a table of values.*

Then we have the following asymptotic formulas defining the intersection point of highway with the boundaries of the permissible region D:

$$\tau_0 = \frac{1}{\lambda\beta}\ln\frac{\lambda k_0^\beta - a_0(1-u_i)}{a_0(u_i - \beta)},$$

$$\sigma_T = \frac{1}{\lambda\beta}\ln\frac{\lambda k_1^\beta - a_0(1-u_i)e^{\rho T}}{a_0(\alpha - (1-u_1)e^{\rho T})}.$$

At that itself highway is determined from the (4.5.15), i.e. is taken limit value k_μ at $\mu \to 0$:

$$k_0^* = \left(\frac{a_0\alpha}{\lambda}\right)^{\frac{1}{\beta}},$$

where $a_0 = b(1-a)$, $\lambda = \varepsilon + n$. It should be noted that the first term in formulas (4.5.23), (4.5.25) are, respectively, the left and right "borderline functions" [15], which approximate the transition from the initial state to the highway and go to the highway in the final state.

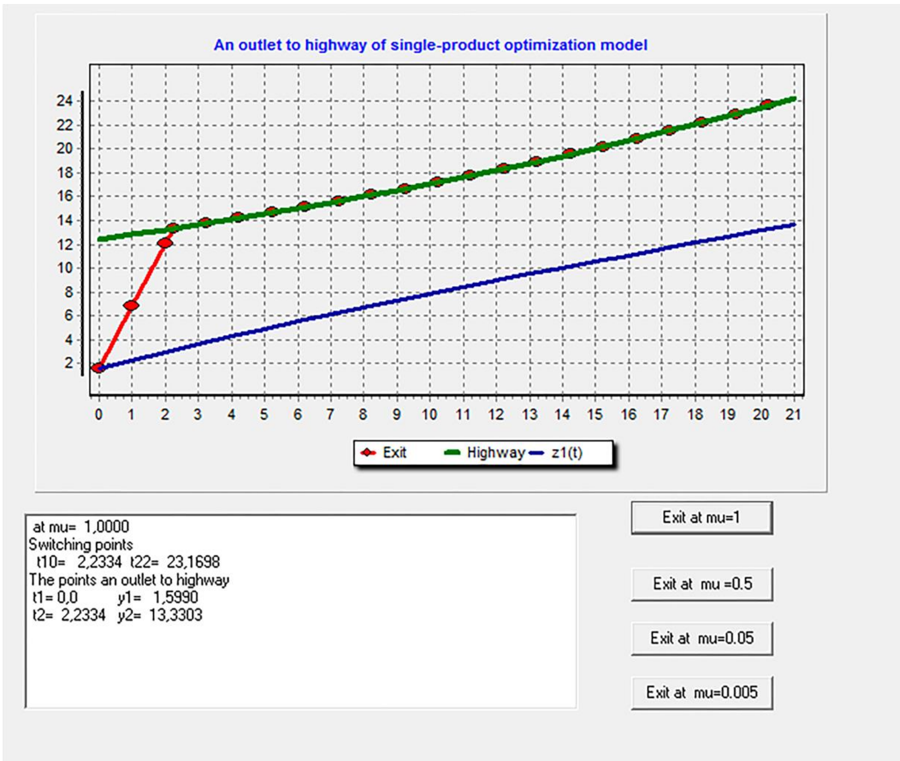


Fig. 4.5.3 *The case at* $\mu = 1$.

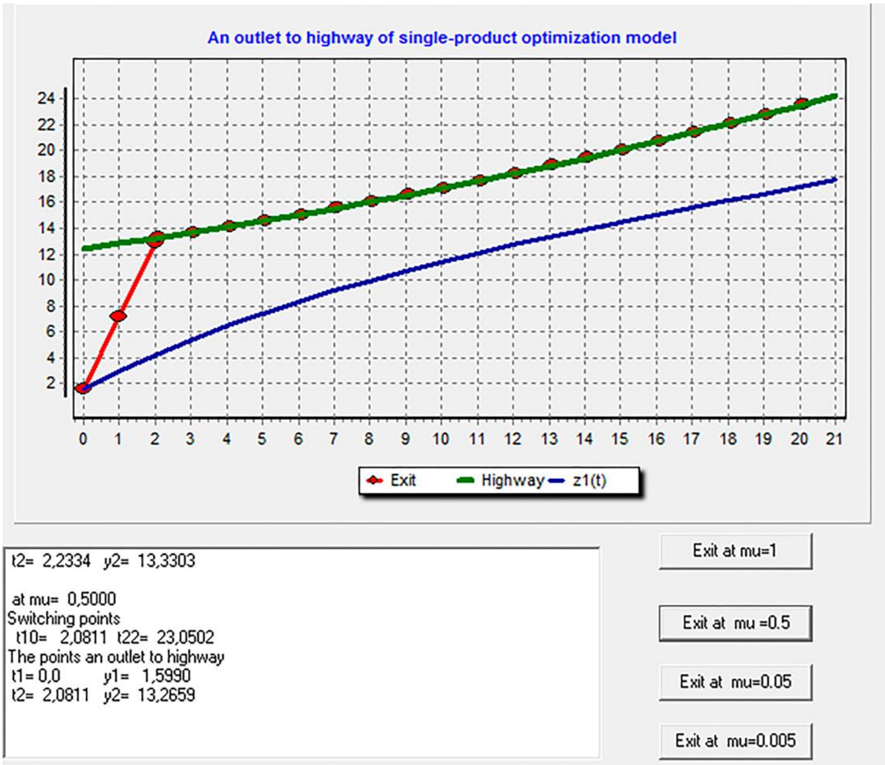


Fig. 4.5.4 *The case at* $\mu = 0.5$.

Note that if for the given problem build the function of Hamilton, then it will depend on the control u linearly and its maximum value are achieved only in the boundary values u. But in real economic problems, as noted in [94, 97], minimum level of consumption is strictly positive: $0 < u_1 \le u \le 1$. Therefore, the Hamiltonian takes the maximum value in points $u = u_1$, $u = 1$ and terms of the values can be determined switching points.

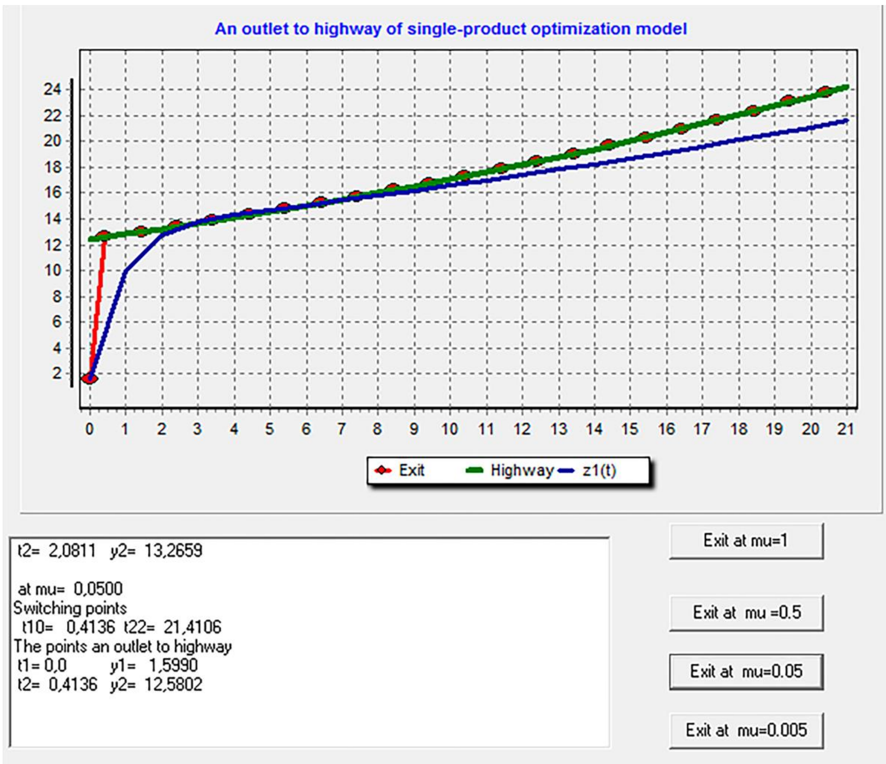


Fig. 4.5.5 *The case at* $\mu = 0.05$.

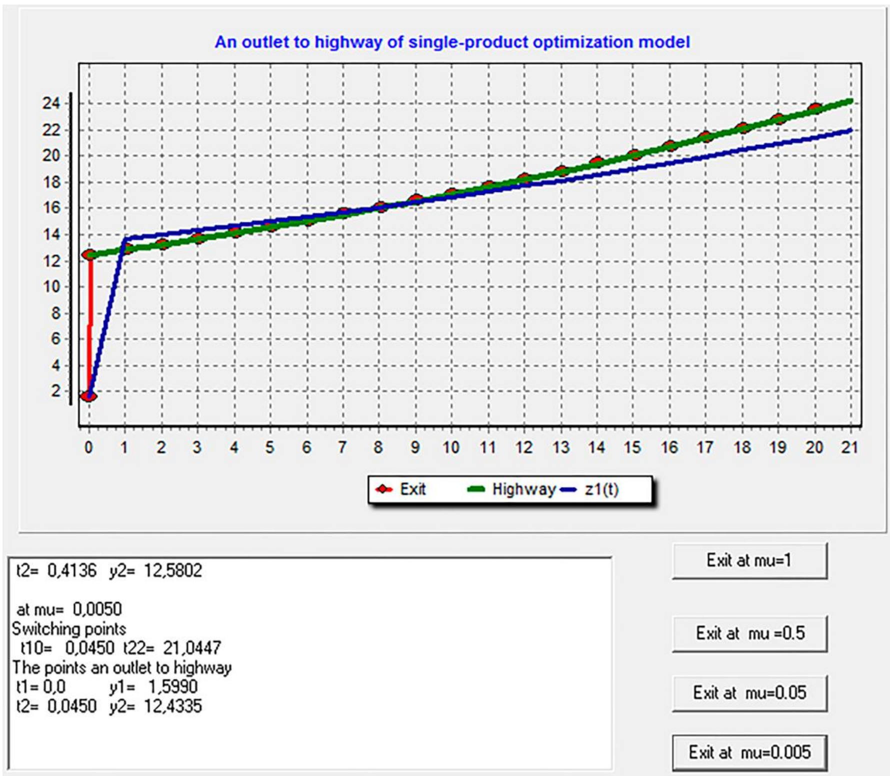
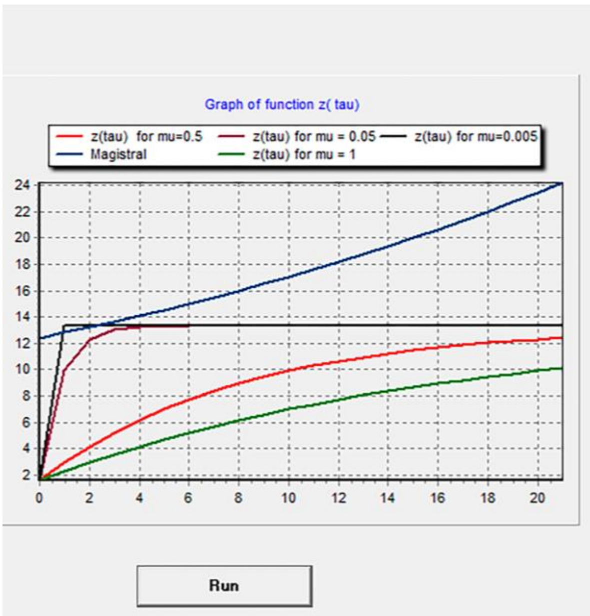


Fig. 4.5.6 *The case at* $\mu = 0.005$.



tau0= 9,0913 sigma0= 9,0311

mu= 0,5000

a1= 4,1230

K0beta = 1,5990 K1beta= 2,7130

t	z1	zz1	ztau
0,0	1,5990	12,4157	1,5990
1,0	2,9788	12,8172	2,9598
2,0	4,2362	13,2317	4,1625
3,0	5,3863	13,6595	5,2255
4,0	6,4426	14,1012	6,1651
5,0	7,4169	14,5572	6,9956
6,0	8,3197	15,0279	7,7296
7,0	9,1603	15,5139	8,3783
8,0	9,9470	16,0155	8,9517
9,0	10,6870	16,5334	9,4585
10,0	11,3870	17,0680	9,9064
11,0	12,0526	17,6199	10,3023
12,0	12,6890	18,1897	10,6522
13,0	13,3007	18,7779	10,9615
14,0	13,8918	19,3851	11,2349
15,0	14,4660	20,0119	11,4765
16,0	15,0263	20,6590	11,6900
17,0	15,5758	21,3271	11,8788
18,0	16,1170	22,0167	12,0456
19,0	16,6521	22,7286	12,1931
20,0	17,1833	23,4636	12,3234
21,0	17,7125	24,2223	12,4386

tau0= 9,0913 sigma0= 9,0311

Fig. 4.5.7 *Graph of function* $z(\tau)$ *and its table of values.*

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Key Words:

Optimal Control, Problem of Moments, Singular Perturbations, Integral Manifolds, Separation Movements, Asymptotic Behavior, Quadratic Functional

Short Introduction to the Book

This book is mainly on the problem of optimal control of singularly perturbed systems and decomposition of variables. The problem of optimal control with respect to the economic processes to which the criterion of control is formulated using the Gram operator of the properties, as well as to deal with the evaluation of the standard deviation of the trajectory of motion of the system. The basic equations and formulas allows to obtain a decision applied to the method of moments, which enables unified computing optimal control of the search procedure and allows separation of variables.

Author's Short Biography

Zamirbek Imanaliev, Professor of the Department of Applied Mathematics and Informatics, Kyrgyz State Technical University after named I.Razzakov. He is engaged in problems of optimal control. He published scholarly works on the separation of variables of the optimal control problem. He mainly works on the theory of optimization and control, mathematical economics and linear programming.

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ISBN 978-1-940366-72-2

9 781940 366722

Price: US $90